மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

M.Sc. MATHEMATICS

II YEAR

CALCULAS OF VARIATIONS AND INTEGRAL EQUATIONS

Sub. Code: SMAM34

Prepared by

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M.Sc. MATHEMATICS – II YEAR

SMAM34: CALCULAS OF VARIATIONS AND INTEGRAL EQUATIONS

SYLLABUS

UNIT-I:

Calculus of Variations and Applications Maxima and Minima – The simplest case – Illustrative example – The variational notation – the more general case.

UNIT-II:

Constraints and Lagrange's Multipliers – Variable endpoints – Sturm Liouville problems – Hamilton's principles – Lagrange equations.

UNIT-III:

Integral Equations – Introduction – Relation between differential and integral equations – The Green's function – Alternative definition of Green's function.

UNIT-IV:

Linear Equations in cause and effect – The influence function – Fredholm equations with separable kernels – Illustrative Examples.

UNIT-V:

Hilbert Schmidt theory – Iterative methods for solving equations of second kind – Fredholm theory.

Recommended Text: Methods of Applied Mathematics

Francis B. Hildebrand, Section 2.1 to 2.11, 3.1 to 3.9 and 3.11

SMAM34 - CALCULUS OF VARIATIONS AND INTEGRAL EQUATIONS

CONTENT

UNIT-I

Calculus of Variation and Application

Definition:

The mapping $x_0 \to f(x_0)$ is a function x_0 is an argument of the function f. At the same time the mapping of a function to the value of the function at a point $f \to f(x_0)$ is functional here x_0 is a parameter.

Section-1: Calculus of variation

The calculus if variation is a branch of mathematics concerned with applying the methods of calculus to finding the maxima and minima of a function which depends for its value or a another function or a curve.

Calculus of variations seek to find to the path, a curve, surface etc for which a given function has a maximum or minimum. Here the functional is defined by

$$I = \int_{x_1}^{x_2} F(x, y, y')\ dx$$

Find the shortest plane curves joining two points (x_1, y_1) and (x_2, y_2).

(i.e)., $\displaystyle\int_{x_1}^{x_2} \sqrt{1 + y'^2}$ is extremum

Application:

The Application of the calculus of variations are concerned with the determination of maxima and minima of certain expressions involving unknown functions.

Condition:

Let $y = f(x)$ be a continuous derivative function defined in the interval (a, b) then, the necessary and sufficient condition for the existence of a maximum or minimum at a point $x = x_0 \in (a, b)$ is

1. $\frac{dy}{dx} = 0$ at $x = x_0$

2. If $y = f(x)$ attains maximum or minimum at $x = x_0$ then $\frac{d^2y}{dx^2} < 0$ at $x = x_0$ or $\frac{d^2y}{dx^2} > 0$ at $x = x_0$

Result:

Let $A = a_{11}x + a_{22}y + a_{33}z + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$

1. To find maxima and minima for two independent variables.

Let z be the function of two independent variables say $z = f(x, y)$ in the region R.

Let the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ exists and are continuous in R. Let $(x_0, y_0) \in R$.

W.K.T., The necessary and sufficient condition for the function z have maxima or minima at the point $(x_0, y_0) \in R$ are $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$.

We can write, these two equations as a single equation that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$ at (x_0, y_0) for arbitrary values of dx and dy.

If $z = f(x, y)$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= f_x dx + f_y dy$$

$$d^2z = d[f_x dx + f_y dy]$$

$$= d[f_x]dx + d[f_y]dy$$

$$= [f_{xx}dx + f_{xy}dy]dx + [f_{yy}dy + f_{yx}dx]dy$$

$$= f_{xx}d^2x + f_{xy}dydx + f_{yy}d^2y + f_{yx}dxdy$$

Hence at (x_0, y_0) the function $z = f(x, y)$ attains its maximum if

$f_{xx} + f_{yy} < 0$

$f_{xx}f_{yy} - f_{xy}^2 < 0$

And it attain its minimum if

$f_{xx} + f_{yy} > 0$

$f_{xx} f_{yy} - f_{xy}^2 > 0$

Note

If $f = f(x_1, x_2, \ldots, x_n)$ be continuously differentiable function of n variables has maximum or minimum value at an interior point of the region $df = 0$ i.e.,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0$$

At the point for all permissible values of the differentials $dx_1, dx_2, \ldots, dx_n$.

<u>Stationary point:</u>

The function $f(x_1, x_2, \ldots, x_n)$ is said to be stationary at a point if $df = 0$

i.e., $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0$

<u>Result-1:</u>

If n variables are all independent, the n differentials can be assign arbitrarily and if follows that $df = 0 \Rightarrow \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0$ is equivalent to n condition $\frac{\partial f}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0$

<u>Result-2:</u>

If n variable are not independent but are related by say N conditions each of the form $\phi_k(x_1, x_2, \ldots, x_n), k = 1,2, \ldots, N$. These N equations can be solved and we can express these N variables in terms the remaining $n - N$ variables. Hence we can express f and df in terms of $n - N$independent variables and differentials.

<u>Lagrange's Multiplier:</u>

Obtain the stationary value of the function $f(x, y, z)$ subject to the constraints

$\phi_1(x, y, z) = 0, \quad \phi_2(x, y, z) = 0$

Proof:

Let f be a stationary value of the function. For the stationary values we have, $df = 0$.

(i.e)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

$$\Rightarrow \; f_x dx + f_y dy + f_z dz = 0 \qquad \_\_\_\_\_(1)$$

Since the three variables x, y, z must satisfy two auxiliary conditions

$$\phi_1(x, y, z) = 0 \qquad \_\_\_\_\_\_(2)$$

$$\phi_2(x, y, z) = 0 \qquad \_\_\_\_\_\_(3)$$

Consider only one variable can be independent from (2) and (3) we ger,

$$\frac{\partial \phi_1}{\partial x}dx + \frac{\partial \phi_1}{\partial y}dy + \frac{\partial \phi_1}{\partial z}dz = 0$$

(i.e).,

$$\phi_{1x}dx + \phi_{1y}dy + \phi_{1z}dz = 0 \qquad \_\_\_\_\_\_(4)$$

$$\frac{\partial \phi_2}{\partial x}dx + \frac{\partial \phi_2}{\partial y}dy + \frac{\partial \phi_2}{\partial z}dz = 0$$

(i.e).,

$$\phi_{2x}dx + \phi_{2y}dy + \phi_{2z}dz = 0 \qquad \_\_\_\_\_\_(5)$$

By solving (4) and (5)

dx and dy can be written in terms of dz.

Then, Substituting dx and dy in equation (1) we get,

$$df = (\quad)dz = 0$$

Since dz can be assigned arbitrarily, the vanishing of the indicated expression in parenthesis in this form is a desired condition that f be stationary when equation (2) and (3) are satisfied.

Another method:

Multiply equation (4) by and equation (5) by λ_2 we get,

$$\lambda_1\phi_{1x}dx + \lambda_1\phi_{1y}dy + \lambda_1\phi_{1z}dz = 0 \quad \_\_\_\_\_\_(6)$$

$$\lambda_2\phi_{2x}dx + \lambda_2\phi_{2y}dy + \lambda_2\phi_{2z}dz = 0 \quad \_\_\_\_\_\_(7)$$

Adding (1), (6), (7),

$$(f_x + \lambda_1\phi_{1x} + \lambda_2\phi_{2x})dx + (f_y + \lambda_1\phi_{1y} + \lambda_2\phi_{2y})dy + (f_z + \lambda_1\phi_{1z} + \lambda_2\phi_{2z})dz = 0 \_\_(8)$$

Arbitrary values of λ_1 and λ_2

Let λ_1 and λ_2 be determined so that two of the parenthesis in equation (8) vanish.

Then, the differential multiplying the remaining parenthesis can be arbitrarily assigned and also vanish,

Thus, we have,

$$f_x + \lambda_1\phi_{1x} + \lambda_2\phi_{2x} = 0 \quad \_\_\_\_\_\_\_\_\_\_(9)$$

$$f_y + \lambda_1\phi_{1y} + \lambda_2\phi_{2y} = 0 \quad \_\_\_\_\_\_\_\_\_\_(10)$$

$$f_z + \lambda_1\phi_{1z} + \lambda_2\phi_{2z} = 0 \quad \_\_\_\_\_\_\_\_\_\_(11)$$

The five equations namely (2), (3), (9), (10), (11) to determine x, y, z, λ_1 and λ_2. The quantities λ_1 and λ_2 are known as Lagrange multiplier.

<u>Note:</u>

1. Lagrange multiplier simplifies the problem also they have physical significance. This procedure is applicable to the general case of n variables and $N < n$ constraints.
2. The conditions (9), (10), (11) are $f + \lambda\phi_1 + \lambda\phi_2$ be stationary when no constraints are present.

1. **Determine the point on the curve of the intersection of the surfaces $z = xy + 5$, $x + y + z = 1$ which is nearest to the origin**

   Let (x, y, z) be the point nearest to the origin, that is, (x, y, z) is a point of intersection of two surfaces $z = xy + 5,\ x + y + z = 1$

We have to minimize

$$F = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$
$$= \sqrt{x^2 + y^2 + z^2}$$

Therefore, $f = x^2 + y^2 + z^2$ subject to the two constraints

$$\phi_1 = z - xy - 5 \_\_\_\_\_\_\_(1)$$

$$\phi_2 = x + y + z - 1 \_\_\_\_\_\_\_(2)$$

Since f is minimum, $df = 0$

$$df = 2xdx + 2ydy + 2zdz = 0$$

$$xdx + ydy + zdz = 0 \_\_\_\_\_\_\_\_\_\_(3)$$

$$\frac{\partial \phi_1}{\partial x}dx + \frac{\partial \phi_1}{\partial y}dy + \frac{\partial \phi_1}{\partial z}dz = 0$$
$$-ydx - xdy + dz = 0 \_\_\_\_\_\_\_\_\_(4)$$

$$\frac{\partial \phi_2}{\partial x}dx + \frac{\partial \phi_2}{\partial y}dy + \frac{\partial \phi_2}{\partial z}dz = 0$$
$$dx + dy + dz = 0 \_\_\_\_\_\_\_\_(5)$$

Choose the multiplier λ_1 and λ_2 and multiply with the equations (4) and (5)

$$(4) \Rightarrow -y\lambda_1 dx - x\lambda_1 dy + \lambda_1 dz = 0 \_\_\_\_\_\_\_(6)$$

$$(5) \Rightarrow \quad \lambda_2 dx + \lambda_2 dy + \lambda_2 dz = 0 \quad \_\_\_\_\_\_\_(7)$$

Adding equations (3), (6) and (7), we get,

$$(x - y\lambda_1 + \lambda_2)dx + (y - x\lambda_1 + \lambda_2)dy + (z + \lambda_1 + \lambda_2)dz = 0$$

$$x - y\lambda_1 + \lambda_2 = 0 \_\_\_\_\_\_\_(a)$$

$$y - x\lambda_1 + \lambda_2 = 0 \_\_\_\_\_\_\_(b)$$

$$z + \lambda_1 + \lambda_2 = 0 \_\_\_\_\_\_\_\_\_(c)$$

Eliminating λ_1 and λ_2 from (a), (b), and (c)

$$\begin{vmatrix} x & -y & 1 \\ y & -x & 1 \\ z & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -y & x \\ 1 & -x & y \\ 1 & 1 & z \end{vmatrix} = 0 \quad (c_1 \leftrightarrow c_3)$$

$$\Rightarrow \begin{vmatrix} 1 & -y & x \\ 0 & -x+y & y-x \\ 0 & 1+y & z-x \end{vmatrix} = 0 \;\; (R_2 \to R_2 - R_1\,; R_3 \to R_3 - R_1)$$

$$\Rightarrow 1[(-x+y)(z-x) - (1+y)(y-x)] = 0$$

$$\Rightarrow [-xz + x^2 + yz - yx - y + x - y^2 + xy] = 0$$

$$\Rightarrow x^2 - y^2 - z(x-y) + (x-y) = 0$$

$$\Rightarrow (x+y)(x-y) - [(x-y)(z-1)] = 0$$

$$\Rightarrow (x-y)[x+y-z+1] = 0$$

$$\Rightarrow x - y = 0 \;\; (or) \;\; (x+y-z+1) = 0$$

Now

$$(1) \Rightarrow z - xy - 5 = 0$$

$$(2) \Rightarrow x + y + z - 1 = 0$$

$$x + y - z + 1 = 0 \; \_\_\_\_\_\_\_\_ (d)$$

$$(2) \Rightarrow x + y = 1 - z$$

$$(d) \Rightarrow 1 - z - z + 1 = 0$$

$$\Rightarrow -2z + 2 = 0$$

$$\Rightarrow z = 1$$

$$(2) \Rightarrow x + y = 0$$

$$(1) \Rightarrow -xy - 4 = 0$$

$$\Rightarrow xy = -4$$

$$\Rightarrow y = -\frac{4}{x}$$

$$(2) \Rightarrow x - \frac{4}{x} = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow x = \pm 2$$

When $x = 2, y = -\frac{4}{2} = -2$

When $x = -2, y = -\frac{4}{-2} = 2$

The points $x = 2,\ y = \ -2\ ;\quad x = -2, y = 2$

Therefore, The points are $(2, -2,1)$ and $(-2,2,1)$

The distance for the 1st point $(2, -2,1)$

$$\Rightarrow \sqrt{(2-0)^2 + (-2-0)^2 + (1-0)^2}$$

$$\Rightarrow \sqrt{4+4+1} = 3$$

The distance for the 2nd point $(-2,2,1)$

$$\Rightarrow \sqrt{(-2-0)^2 + (2-0)^2 + (1-0)^2}$$

$$\Rightarrow \sqrt{4+4+1} = 3$$

The required point is 3units distance from the origin

$$(1) \Rightarrow z - xy - 5 = 0$$

$$(2) \Rightarrow x - y + z - 1 = 0$$

$$\Rightarrow x - y = 0 \_\_\_\_\_\_\_\_\_(e)$$

$$\Rightarrow x = y$$

$$(2) \Rightarrow 2y + z - 1 = 0$$

$$\Rightarrow z = 1 - 2y$$

$$(1) \Rightarrow z - y^2 - 5 = 0$$

$$\Rightarrow (1 - 2y) - y^2 - 5 = 0$$

$$\Rightarrow \ -y^2 - 2y - 4 = 0$$

$$\Rightarrow y^2 + 2y + 4 = 0$$

$$y = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$y = \frac{-2 \pm i2\sqrt{3}}{2}$$

$$y = -1 \pm i\sqrt{3}$$

The equation have no real common solution. Hence $(2, -2,1)$ and $(-2,2,1)$ are the points on the curve of intersection of surfaces nearest to the origin.

2. **Find the points on central quadratic surface $\boldsymbol{\phi = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz = constant}$ for which the distance from the origin is maximum or minimum relative to neighbouring points.**

To find the stationary value of f

$$f = (x - 0)^2 + (y - 0)^2 + (z - 0)^2$$
$$f = x^2 + y^2 + z^2$$

Since f is stationary, $df = 0$

$$2xdx + 2ydy + 2zdz = 0$$

$$\Rightarrow xdx + ydy + zdz = 0 \_\_\_\_\_\_\_\_\_\_ (1)$$

Given constant $\phi = constant$

$$\phi = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz$$

$$d\phi = (2a_{11}x + 2a_{12}y + 2a_{13}z\ )dx + (2a_{22}y + 2a_{12}x + 2a_{23}z)dy$$

$$+(2a_{33}z + 2a_{23}y + 2a_{13}x)dz \quad \underline{\hspace{2cm}}(A)$$

Choose the multiplier λ_1 and multiply with (A) we ger,

$$2\lambda_1(a_{11}x + a_{12}y + a_{13}z\,)dx + 2\lambda_1(a_{22}y + a_{12}x + a_{23}z)dy$$

$$+2\lambda_1(a_{33}z + a_{23}y + a_{13}x)dz = 0 \underline{\hspace{2cm}}(2)$$

Adding (1) and (2) we get,

$$[x + 2\lambda_1(a_{11}x + a_{12}y + a_{13}z\,)]dx + [y + 2\lambda_1(a_{22}y + a_{12}x + a_{23}z)]dy$$

$$+[z + 2\lambda_1(a_{33}z + a_{23}y + a_{13}x)]dz = 0$$

$$x + 2\lambda_1(a_{11}x + a_{12}y + a_{13}z\,) = 0$$

$$\Rightarrow \lambda_1 = \frac{-x}{2(a_{11}x + a_{12}y + a_{13}z\,)}$$

$$y + 2\lambda_1(a_{22}y + a_{12}x + a_{23}z) = 0$$

$$\Rightarrow \lambda_1 = \frac{-y}{2(a_{22}y + a_{12}x + a_{23}z)}$$

$$z + 2\lambda_1(a_{33}z + a_{23}y + a_{13}x) = 0$$

$$\Rightarrow \lambda_1 = \frac{z}{2(a_{33}z + a_{23}y + a_{13}x)}$$

These equations determine the stationary point (x, y, z) on the given surface the values of λ for which the system of equation has a non-trivial solution (x, y, z) are known as the characteristics values of λ. These solutions are the points on the surface which are at minimal distance from the origin.

SIMPLEST CASE

1. Derive Euler's equation.

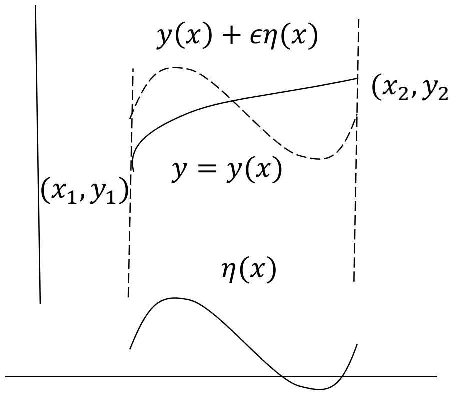
(or)

Determine the condition $y = y(x)$ to maximise or minimise for the integral $I = \int_{x_1}^{x_2} F(x, y, y')dx$

(or)

Find a continuously differentiable function $y = y(x)$ for which the integral $I = \int_{x_1}^{x_2} F(x, y, y')dx$ takes on a maximum or minimum value and which satisfies the end condition $y(x_1) = y_1$ and $y(x_2) = y_2$

Proof



Let $y = y(x)$ be a curve joining the points (x_1, y_1) and (x_2, y_2)

$I = \int_{x_1}^{x_2} F(x, y, y')dx$ ________(1) is extremum.

$y = y(x) + \epsilon\eta(x)$ be a neighbouring curve joining the points (x_1, y_1) and (x_2, y_2), ϵ is a parameter and $\eta(x)$ is arbitrary function. $\eta(x_1) = 0$ and $\eta(x_2) = 0$

$$I = \int_{x_1}^{x_2} F\big(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\big)dx \quad \text{\_\_\_\_\_\_\_\_}(A)$$

After the simplification, we get I is the function with respect to ϵ

Since I is extremum, $\frac{dI}{d\epsilon} = 0$

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial x}\left(\frac{\partial x}{\partial \epsilon}\right) + \frac{\partial F}{\partial y}\left(\frac{\partial y}{\partial \epsilon}\right) + \frac{\partial F}{\partial y'}\left(\frac{\partial y'}{\partial \epsilon}\right)\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx + \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx + \frac{\partial f}{\partial y'}[\eta(x_2) - \eta(x_1)] - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$0 = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Hence if $y(x)$ minimize or maximise the integral $I = \int_{x_1}^{x_2} F(x, y, y')dx$ must satisfied the Euler's equation.

Therefore $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

Definition (Euler equation):

The condition for $y = y(x)$ to maximise or minimise the integral $I = \int_{x_1}^{x_2} F(x, y, y')dx$ is given by the Euler equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

Another form of Euler Equation:

Euler Equation: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ ____________(1)

Since $\frac{\partial f}{\partial y'}$ is a function of x, y and $y' = \frac{dy}{dx}$

$$(1) \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)\frac{dx}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)\frac{dy'}{dx} - \frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)y'' - \frac{\partial F}{\partial y} = 0$$

$$\Rightarrow F_{y'x} + F_{y'y}y' + F_{y'y'}y'' - F_y = 0$$

Note -1:

The three forms of Euler Equation,

1. $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$
2. $F_{y'y'}\frac{d^2y}{dx^2} + F_{y'y}\frac{dy}{dx} + F_{y'x} - F_y = 0$
3. $\frac{1}{y\prime}\left(\frac{d}{dx}\left(F - \frac{\partial f}{\partial y'}\frac{dy}{dx}\right) - \frac{\partial F}{\partial x}\right) = 0$

Note -2:

1. F is independent of x in Euler equation, then $\frac{\partial F}{\partial x} = 0$

   Euler Equation: $\frac{1}{y\prime}\left(\frac{d}{dx}\left(F - \frac{\partial f}{\partial y'}\frac{dy}{dx}\right) - \frac{\partial F}{\partial x}\right) = 0$

   $$\Rightarrow \frac{d}{dx}\left(F - \frac{\partial f}{\partial y'}\frac{dy}{dx}\right) = 0$$

   $$\Rightarrow F - \frac{\partial f}{\partial y'}\frac{dy}{dx} = constant$$

2. F is independent of $y, \frac{\partial F}{\partial y} = 0$

   Euler equation: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

   $$\Rightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

   $$\frac{\partial f}{\partial y'} = constant$$

3. F is independent of $y', \frac{\partial f}{\partial y'} = 0$

   Euler equation: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

   $$\Rightarrow \frac{\partial F}{\partial y} = 0$$

Definition:

The solution of the Euler equations are known as **extremals** of the given problem

Definition:

An extremal which satisfies the appropriate the end condition is called **stationary function** of the problem.

1. **Show that the shortest distance between two points in a plane is a straight line.**

   Let the points in the plane be $(x_1, y_1), (x_2, y_2)$

   To minimise the curve $y = y(x)$

   To minimise the integral $I = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\ dx$

   W. K. T., $I = \int_{x_1}^{x_2} F(x, y, y')dx$

   Here $F = \sqrt{1 + y'^2}$

   Euler equation,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}(1 + y'^2)^{-\frac{1}{2}} * (2y')\right) - 0 = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\Rightarrow \frac{\sqrt{1 + y'^2}\, y'' - y' \frac{1}{2}(1 + y'^2)^{-\frac{1}{2}} (2y')y''}{\left(\sqrt{1 + y'^2}\right)^2} = 0$$

$$\Rightarrow \sqrt{1 + y'^2}\ \ y'' - \frac{y'^2 y''}{\sqrt{1 + y'^2}} = 0$$

$$\Rightarrow \frac{(1 + y'^2)y'' - y'^2 y''}{\sqrt{1 + y'^2}} = 0$$

$$\Rightarrow y'' + y'^2 y'' - y'^2 y'' = 0$$

$$\Rightarrow y'' = 0$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 0$$

$$Integrating \ \Rightarrow \frac{dy}{dx} = a$$

$$Integrating \Rightarrow y(x) = ax + b$$

2nd Method:

Here F is independent of x, then

$$F - \frac{\partial F}{\partial y'} y' = constant$$

$$\sqrt{1 + y'^2} - \frac{1}{2}(1 + y'^2)^{-\frac{1}{2}}.2y'.y' = constant$$

$$\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}} = constant$$

$$\frac{(1 + y'^2) - y'^2}{\sqrt{1 + y'^2}} = constant$$

$$\Rightarrow \frac{1}{\sqrt{1 + y'^2}} = \frac{1}{k}$$

$$\Rightarrow \sqrt{1 + y'^2} = k$$

Squaring on both sides

$$\Rightarrow 1 + y'^2 = k^2$$

$$\Rightarrow y'^2 = k^2 - 1$$

$$\Rightarrow y' = \sqrt{k^2 - 1} = a$$

$$\Rightarrow y' = a$$

$$\Rightarrow \frac{dy}{dx} = a$$

$$Integrating \Rightarrow y(x) = ax + b$$

3rd Method:

Here F is independent of y, then

$$\frac{\partial F}{\partial y'} = constant$$

$$\frac{1}{2}(1+y'^2)^{-\frac{1}{2}}(2y') = constant$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = constant$$

$$\Rightarrow y' = c\sqrt{1+y'^2}$$

Squaring on both sides

$$y'^2 = c^2(1+y'^2)$$

$$y'^2 = c^2 + c^2y'^2$$

$$y'^2(1-c^2) = c^2$$

$$y'^2 = \frac{c^2}{1-c^2}$$

$$y' = \frac{c}{\sqrt{1-c^2}}$$

$$\frac{dy}{dx} = a$$

$$Integrating, y(x) = ax + b$$

2. **S. T the minimal surface of revolution passing through two given points be catenary.**

   Let P and Q be two points in a plane.

   P and Q can be joined by infinite number of curves each curve will generate a surface, where it is rotated through 2π about the x −axis.

   Here to find the curve which will give the minimum surface area

$$I = \int_{x_1}^{x_2} F(x, y, y')dx$$

$$I = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}dx$$

$$\frac{I}{2\pi} = \int_{x_1}^{x_2} y\sqrt{1+y'^2}dx$$

Here $F = y\sqrt{1+y'^2}$

Euler equation: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

$$\frac{d}{dx}\left[y.\frac{1}{2}(1+y'^2)^{-\frac{1}{2}}.2y'\right] - \sqrt{1+y'^2} = 0$$

$$\frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] - \sqrt{1+y'^2} = 0$$

$$\frac{\sqrt{1+y'^2}(yy''+y'y') - \frac{yy'1}{2}(1+y'^2)^{-\frac{1}{2}}2y'y''}{\left(\sqrt{1+y'^2}\right)^2} - \sqrt{1+y'^2} = 0$$

$$\frac{\sqrt{1+y'^2}(yy''+y'^2) - \frac{yy'^2y''}{\sqrt{1+y'^2}}}{(1+y'^2)} - \sqrt{1+y'^2} = 0$$

$$\frac{(1+y'^2)(yy''+y'^2) - yy'^2y''}{\sqrt{1+y'^2}(1+y'^2)} - \sqrt{1+y'^2} = 0$$

$$yy'' + y'^2 + yy''y'^2 + y'^4 - yy'^2y'' - (1+y'^2)^2 = 0$$

$$yy'' + y'^2 + y'^4 - 1 - y'^4 - 2y'^2 = 0$$

$$y'' - y'^2 - 1 = 0 \_\_\_\_\_\_\_\_\_\_(A)$$

Let $y' = p$

$$y'' = p = \frac{dp}{dx} = \frac{dp}{dy}.\frac{dy}{dx}$$

$$y'' = p\frac{dp}{dy}$$

$$(A) \Rightarrow yp\frac{dp}{dy} - p^2 - 1 = 0$$

$$yp\frac{dp}{dy} = 1 + p^2$$

$$\frac{pdp}{1+p^2} = \frac{dy}{y}$$

Integrating,

$$\frac{1}{2}\log(1+p^2) = \log y + \log c$$

$$\log(1+p^2)^{\frac{1}{2}} - \log c = \log y$$

$$\log\left(\frac{\sqrt{1+p^2}}{c}\right) = \log y$$

$$\frac{\sqrt{1+p^2}}{c} = y$$

$$y = a\sqrt{1+p^2}, where \frac{1}{c} = a$$

$$y^2 = a^2(1+p^2)$$

$$p^2 = \frac{y^2 - a^2}{a^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2 - a^2}{a^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - a^2}}{a}$$

$$\frac{dy}{\sqrt{y^2 - a^2}} = \frac{dx}{a}$$

$$\cosh^{-1}\left(\frac{y}{a}\right) = \frac{x}{a} + b$$

$$\frac{y}{a} = cosh\left(\frac{x}{a} + b\right)$$

$$y = a\ cosh\left(\frac{x}{a} + b\right)$$

2nd Method:

Euler equation: $F - \frac{\partial F}{\partial y'} y' = constant$

$$y\sqrt{1+y'^2} - \frac{1}{2}(1+y'^2)^{-\frac{1}{2}}.2y'.y' = constant$$

$$y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = c$$

$$\frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c$$

$$\frac{y}{\sqrt{1+y'^2}} = c$$

$$y = c\sqrt{1+y'^2}$$

$$y^2 = c^2(1+y'^2)$$

$$y^2 = c^2 + c^2y'^2$$

$$y^2 - c^2 = c^2\left(\frac{dy}{dx}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2 - c^2}{c^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c}$$

Integrating,

$$\cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + b$$

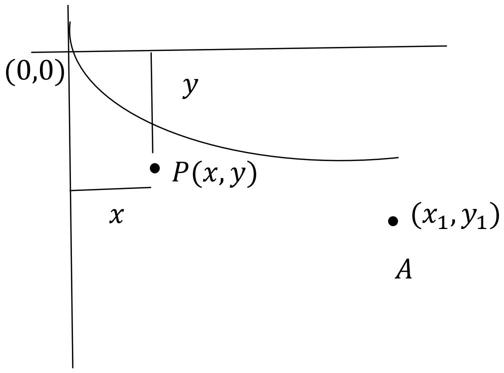
$$\frac{y}{c} = cosh\left(\frac{x}{c} + b\right)$$

$$y = c\ cosh\left(\frac{x}{c} + b\right)$$

Brachisto Chrome Problem:

(or)

Find the shortest time of the curve



Suppose a particle starts from rest at a point $(0,0)$ and slides down the curve at OA.

Let $P(x,y)$ be the position of the particle at time t. If v is the velocity of the particle at a time t.

$$v = \sqrt{2gy}$$

$$\frac{ds}{dt} = \sqrt{2gy}$$

$$ds = \sqrt{2gy}dt$$

W. K.T., $ds = \sqrt{1+y'^2}\,dx$

$$\sqrt{1+y'^2}dx = \sqrt{2gy}dt$$

$$dt = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

$$dt = \frac{1}{\sqrt{2g}}\left(\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right)dx$$

$$\int_0^{x_1} dt = \frac{1}{\sqrt{2g}}\int_0^{x_1}\left(\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right)dx$$

$$T = \int_0^{x_1}\left(\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right)dx$$

In Euler equation,

$$I = \int_{x_1}^{x_2} F(x,y,y')dx$$

Here $F(x,y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$

Here F is independent of x

$$F - \frac{\partial F}{\partial y'}y' = constant$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{1}{\sqrt{y}}\frac{1}{2}(1+y'^2)^{-\frac{1}{2}}.2y'y' = c$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$\frac{1+y'^2-y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$\sqrt{y}\sqrt{1+y'^2} = c$$

Put $y' = \cot\theta$

$$\sqrt{y}\sqrt{1+y'^2} = c$$

$$\sqrt{y}\sqrt{1+\cot^2\theta} = c$$

$$\sqrt{y}\sqrt{cosec^2\,\theta} = c$$

$$\sqrt{y} = \frac{c}{\sqrt{cosec^2\,\theta}}$$

$$\sqrt{y} = c\sin\theta$$

$$y = c^2\sin^2\theta$$

$$y = c^2\left(\frac{1-\cos 2\theta}{2}\right)$$

$$y = a(1-\cos 2\theta)$$

Now,

$$\frac{dx}{d\theta} = \frac{dx}{dy}\frac{dy}{d\theta}$$

$$= \frac{1}{y'}a(2\sin 2\theta\ )$$

$$= \frac{1}{\cot\theta}2a\sin 2\theta$$

$$= \frac{2a\ (2\sin\theta\cos\theta)}{\frac{\cos\theta}{\sin\theta}}$$

$$= 4a\sin^2\theta$$

$$= 4a\left(\frac{1-\cos 2\theta}{2}\right)$$

$$= 2a(1-\cos 2\theta)$$

Integrating,

$$x = 2a\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$= 2a\left(\frac{2\theta - \sin 2\theta}{2}\right)$$

$$= a(2\theta - \sin 2\theta)$$

The required curve is $y = a(1 - \cos 2\theta), x = a(2\theta - \sin 2\theta)$ which is the equation of cycloid.

Natural Boundary Conditions and Transition Conditions:

The value of unknown function $y(x)$ is not preassigned at one or both of the end points x_1, x_2, the difference $\epsilon\eta(x)$ between the true function $y = y(x)$ and the variate function $y = y(x) + \epsilon\eta(x)$ need not vanish.

In Euler equation derivation, we used $\eta(x_1) = 0$ and $\eta(x_2) = 0$, but here, the variation is such that $(\eta(x_1) \neq 0 \; or \; \eta(x_2) \neq 0) \;\; or \; (\eta(x_1) \neq 0 \; and \;\; \eta(x_2) \neq 0)$

The equation is

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x)dx + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_1}^{x_2} = 0$$

This is true all permissible variation $\epsilon\eta(x)$. It must vanish for all variations which are zero at both ends. i.e., $\eta(x_1) = 0 \; and \;\; \eta(x_2) = 0$

We get,

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x)dx = 0$$

Therefore, we yield an Euler equation.

Note:

1. Now,

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_1}^{x_2} = 0$$

$$\frac{\partial F}{\partial y'}\eta(x_2) - \frac{\partial F}{\partial y'}\eta(x_1) = 0$$

Since $\eta(x_1)$ and $\eta(x_2)$ are arbitrary, the co-efficient must vanish.

$$\left(\frac{\partial F}{\partial y'}\right)_{x=x_1} = 0, \qquad \left(\frac{\partial F}{\partial y'}\right)_{x=x_2} = 0$$

If $y(x_1)$ is not given, then $\left(\frac{\partial F}{\partial y'}\right)_{x=x_1} = 0$

If $y(x_2)$ is not given, then $\left(\frac{\partial F}{\partial y'}\right)_{x=x_2} = 0$

Are called the Natural Boundary Condition of the problem.

2. If $y(x_1)$ is preassigned as y_1, whereas $y(x_2)$ is not given then the boundary condition is $y(x_1) = y_1$ and $\left(\frac{\partial F}{\partial y'}\right)_{x=x_2} = 0$

   Similarly, other boundary condition is $y(x_2) = y_2$ and $\left(\frac{\partial F}{\partial y'}\right)_{x=x_1} = 0$

Euler Equation:

When $\frac{\partial F}{\partial y}$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ are discontinuous at one or more points inside the integral.

Consider $I = \int_{x_1}^{x_2} F(x, y, y')dx$ where F may be such that any one or both the terms of $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ and $\frac{\partial F}{\partial y}$ are discontinuous at one or more points inside the integral.

Assume that there is only one point of discontinuity at $x = c$ then

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x)dx + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_1}^{x_2} = 0$$

Taken the form

$$\int_{x_1}^{c^-} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x)dx + \int_{c^+}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x)dx + + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_1}^{c^-} + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{c^+}^{x_2} = 0$$

Assume that the minimizing function $y = y(x)$ is continuous at $x = c$, all the admissible function $y(x) + \epsilon\eta(x)$ have the same property.

Therefore, we have $\eta(c^+) = \eta(c^-) = \eta(c)$

$$\int_{x_1}^{c^-}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx+\int_{c^+}^{x_2}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx$$
$$+\left[\left(\frac{\partial F}{\partial y'}\right)_{x=x_2}\eta(x_2)-\left(\frac{\partial F}{\partial y'}\right)_{x=c^+}\eta(c^+)\right]=0$$

$$\int_{x_1}^{c^-}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx+\int_{c^+}^{x_2}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx$$
$$+\left[\left(\frac{\partial F}{\partial y'}\right)_{x=x_2}\eta(x_2)-\left(\frac{\partial F}{\partial y'}\right)_{x=x_1}\eta(x_1)\right]$$
$$-\left[\left(\frac{\partial F}{\partial y'}\right)_{x=c^+}\eta(c^+)-\left(\frac{\partial F}{\partial y'}\right)_{x=c^-}\eta(c^-)\right]=0$$

$$\int_{x_1}^{c^-}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx+\int_{c^+}^{x_2}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx$$
$$+\left[\left(\frac{\partial F}{\partial y'}\right)_{x=x_2}\eta(x_2)-\left(\frac{\partial F}{\partial y'}\right)_{x=x_1}\eta(x_1)\right]-\left[\left(\frac{\partial F}{\partial y'}\right)_{x=c^+}-\left(\frac{\partial F}{\partial y'}\right)_{x=c^-}\right]=0$$

Hence Euler equation must hold in each of the subintervals (x_1, c) and (c, x_2) if $\frac{\partial F}{\partial y'}$ vanish at any end points $x = x_1$ or $x = x_2$ where, y is not prescribe and also the natural transaction condition called $y(c^+) = y(c^-)$ _______(a)

i.e., $\lim_{x\to c^+}\frac{\partial F}{\partial y'} = \lim_{x\to c^-}\frac{\partial F}{\partial y'}$ _____(b) must be satisfied at the point $x = c$

Note:

1. The condition (a) says that y is continuous at $x = c$
2. The condition (b) says that the derivative y' is discontinuous at that point.

1. **Determine the stationary function associated with the integral $I = \int_0^1 (Ty'^2 - \rho^2\omega^2y^2)dx$ where T, ρ, ω are given constant.**

Given $I = \int_0^1 (Ty'^2 - \rho^2\omega^2y^2)dx$

Here $F = Ty'^2 - \rho^2\omega^2y^2$

To find: The stationary function it must satisfy the Euler equation.

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

$$\frac{d}{dx}(2Ty') - (-2\rho^2\omega^2 y) = 0$$

$$2Ty'' + 2\rho^2\omega^2 y = 0$$

$$Ty'' + \rho^2\omega^2 y = 0$$

$$y'' = -\frac{\rho^2\omega^2 y}{T}$$

$y'' = -\alpha^2 y$ where $\alpha^2 = \frac{\rho^2\omega^2}{T}$

$$y'' + \alpha^2 y = 0$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

$$(D^2 + \alpha^2)y = 0$$

The auxiliary equation is $m^2 + \alpha^2 = 0$

$$m^2 = -\alpha^2$$

$$m = \pm i\alpha$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x \;\_\_\_\_\_\_\_\_\_\_\_(1)$$

Case-1:

Let the condition $y(0) = 0, y(1) = 1$ be prescribed

$$(1) \Rightarrow y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y(0) = c_1 \cos 0 + c_2 \sin 0$$

$$c_1 = 0$$

$$y(1) = c_1 \cos \alpha + c_2 \sin \alpha$$

$$1 = 0 + c_2 \sin\alpha$$

$$c_2 = \frac{1}{\sin\alpha}$$

$$y(x) = 0 + \frac{1}{\sin\alpha}\sin\alpha x$$

$$= \frac{\sin\alpha x}{\sin\alpha}\ if\ \alpha \neq 0, \pi, 2\pi, 3\pi, \ldots$$

Case-2:

Let the condition $y(1) = 1$ be assigned but $y(0)$ is not assigned.

If $x = 0$ is not given $\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0$

$$(1) \Rightarrow y(1) = c_1 \cos\alpha + c_2 \sin\alpha$$

$$1 = c_1 \cos\alpha + c_2 \sin\alpha \ \_\_\_\_\_\_\_\_\_\_(2)$$

$$\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0$$

$$(2Ty')_{x=0} = 0, y' = 0\ at\ x = 0$$

$$(1) \Rightarrow y'(x) = -\alpha c_1 \sin\alpha x + \alpha c_2 \cos\alpha x$$

$$y'(0) = -\alpha c_1 \sin 0 + \alpha c_2 \cos 0$$

$$0 = \alpha c_2$$

$$c_2 = 0$$

$$(2) \Rightarrow 1 = c_1 \cos\alpha + 0$$

$$c_1 = \frac{1}{\cos\alpha}$$

$$(1) \Rightarrow y(x) = \frac{1}{\cos\alpha}\cos\alpha\, x + 0$$

$$= \frac{\cos\alpha\, x}{\cos\alpha}\ if\ \alpha \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Case-3:

Neither $y(0)$ nor $y(1)$ not preassigned.

i.e., $\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0$ and $\left(\frac{\partial F}{\partial y'}\right)_{x=1} = 0$

Now,

$$\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0$$

$$(2Ty')_{x=0} = 0$$

$$y' = 0 \; at \; x = 0$$

$$(1) \Rightarrow y'(x) = -c\alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

$$y'(0) = 0 + c_2 \alpha \cos 0$$

$$\Rightarrow c_2 \alpha = 0$$

$$\Rightarrow c_2 = 0$$

$$\left(\frac{\partial F}{\partial y'}\right)_{x=1} = 0$$

$$(2Ty')_{x=1} = 0$$

$$y' = 0 \; at \; x = 1$$

$$(1) \Rightarrow y'(x) = -c_1 \alpha \sin \alpha x + c_2 \cos \alpha x$$

$$y'(1) = -c_1 \alpha \sin \alpha + c_2 \cos \alpha$$

$$0 = -c_1 \alpha \sin \alpha$$

$$c_1 = 0$$

$$(1) \Rightarrow y(x) = 0$$

$$\Rightarrow y = 0$$

Case-4:

Let F be discontinuous at $x = c$

Let $T = T_1,\ \rho = \rho_1,\ \omega = \omega_1,\ where\ 0 \leq x \leq c$

$T = T_2, \qquad \rho = \rho_2, \qquad \omega = \omega_2, \qquad where\ c < x \leq 1$

Where $T_1,\ T_2, \rho_1, \rho_2, \omega_1, \omega_2$ are positive constants and $y(0) = 0$ and $y(1) = 1$ are given

Now, by the case-1, we have,

$y(x) = c_1 \cos \alpha_1 x + c_2 \sin \alpha_1 x$ for $0 \leq x \leq c$

$y(x) = c_1 \cos \alpha_2 x + c_2 \sin \alpha_2 x$ for $c \leq x \leq 1$ and $\alpha_2^2 = \frac{\rho_2^2 \omega_2^2}{T_2}$

Now, the natural transition conditions are,$y(c^+) = y(c^-)$

$$\Rightarrow \lim_{x \to c^+} \frac{\partial F}{\partial y'} = \lim_{x \to c^-} \frac{\partial F}{\partial y'}$$

$$\Rightarrow 2T_2 \lim_{x \to c^+} y'(x) = 2T_1 \lim_{x \to c^-} y'(x)$$

$$\Rightarrow T_2 \lim_{x \to c^+} y'(x) = T_1 \lim_{x \to c^-} y'(x)$$

These are the equations for natural transition conditions.

<u>The Variational Notation:</u>

Let F be the set of function which satisfy the certain conditions:

Any quantity which assigns a specific numerical value corresponding to each function in $\mathcal{F}$ is said to be **Functional on the set.**

<u>Example:</u>

Define $\mathcal{F} \to V$ the set of all functions of a single variable x which possess the continuous 1^{st} derivative at all points in an interval $a \leq x \leq b$, then the integral

$$I_1 = \int_a^b y(x)dx$$

Is a function on $\mathcal{F}$ and

$$I_2 = \int_a^b \left[y(x)y''(x) - \left(y'(x)\right)^2\right] dx$$

Is also a function on $\mathcal{F}$

Definition:

Consider the integrant of the form $= F(x, y, y')$ F, for a fixed value of x depends upon the function $y = y(x)$ and its derivative.

Let $y(x)$ be a function to be determined while changing $y(x)$ into a new function $y(x) + \epsilon\eta(x)$. The change $\epsilon\eta(x)$ in $y(x)$ is called variation of y and it is denoted by $\delta y = \epsilon\eta(x)$.

Definition:

Corresponding to the change in $y(x)$ for a fixed value of x, the functional changes by an amount ΔF where

$$\Delta F = F(x, y + \epsilon\eta, y' + \epsilon\eta') = F(x, y, y')$$

$$\Delta F = \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta' + terms\ inovling\ higher\ power\ of\ \epsilon$$

The 1st two terms in the above equations are defined to be the **variation of *F***

Therefore,

$$\delta F = \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$$

Note:

1. $F = y$

   $\delta F = \delta y$

   $\delta y = \frac{\partial F}{\partial y}\epsilon\eta(x) + \frac{\partial F}{\partial y'}\epsilon\eta'(x)$

Equating the coefficient of $\epsilon\eta(x)$ and $\epsilon\eta'(x)$ we get,

$$\frac{\partial F}{\partial y} = 1, \frac{\partial F}{\partial y'} = 0$$

Therefore, when $F = y, \delta F = \epsilon\eta(x)$

2. $F = y'$

   $\delta F = \delta y'$

   $\delta y' = \frac{\partial F}{\partial y}\epsilon\eta(x) + \frac{\partial F}{\partial y'}\epsilon\eta'(x)$

   $\epsilon\eta'(x) = \frac{\partial F}{\partial y}\epsilon\eta(x) + \frac{\partial F}{\partial y'}\epsilon\eta'(x)$

Equating the coefficient,

$$\frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial y'} = 1$$

Therefore, when $F = y', \delta F = \epsilon\eta'(x)$

3. Combining the variation of F and y we get,

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

Let $F = F(x, y, y')$

By the definition of differential we have,

$$\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

Since x is fixed, $\delta x = 0$

Therefore,

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

There is an analog between equation (1) and (2).

Note:

The differential of the function is the 1^{st} order approximation to the change in that function along the particular curve while the variational of the function is the 1^{st} order approximation to the change from curve to curve.

Result:

1. $\delta(F_1 + F_2) = \delta F_1 + \delta F_2$
2. $\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1$
3. $\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}$

General Case: [Analog definition]

Let x, y be independent variables and u, v be dependent variable.

Consider the function $F = F(x, y, u, v, u_x, v_x, u_y, v_y)$. Fix x, y and vary both u, v into new function $u + \epsilon\xi, v + \epsilon\eta$ and define the variation as follows:

$$\delta u = \epsilon\xi(x, y), \delta v = \epsilon\eta(x, y)$$

Hence the change in variable is given by,

$$\Delta F = \frac{\partial F}{\partial u}\epsilon\xi + \frac{\partial F}{\partial v}\epsilon\eta + \frac{\partial F}{\partial u_x}\epsilon\xi_x + \frac{\partial F}{\partial v_x}\epsilon\eta_x + \frac{\partial F}{\partial u_y}\epsilon\xi_y + \frac{\partial F}{\partial v_y}\epsilon\eta_y$$
$$+ \, terms \ involving \ higher \ powe \ of \ \epsilon$$

$$\Delta F = \frac{\partial F}{\partial u}\epsilon\xi + \frac{\partial F}{\partial v}\epsilon\eta + \frac{\partial F}{\partial u_x}\epsilon\xi_x + \frac{\partial F}{\partial v_x}\epsilon\eta_x + \frac{\partial F}{\partial u_y}\epsilon\xi_y + \frac{\partial F}{\partial v_y}\epsilon\eta_y$$

$$= \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial v_x}\delta v_X + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v_y}\delta v_y$$

Result-1:

If x is independent variable, then $\delta, \frac{d}{dx}$ are commutative.

(i.e.), $\frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right)$

Proof:

W. K. T., $\delta y = \epsilon\eta(x)$ and $\delta y' = \epsilon\eta'(x)$

$$\frac{d}{dx}(\delta y) = \frac{d}{dx}\big(\epsilon\eta(x)\big)$$

$$= \epsilon \frac{d}{dx}\big(\eta(x)\big)$$

$$= \epsilon\eta'(x)$$

$$= \delta y'$$

$$\delta\left(\frac{dy}{dx}\right)$$

Therefore, δ and $\frac{d}{dx}$ are commutative.

<u>Result-2:</u>

If x and y are independent variable, then $\delta, \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commutative.

(i.e.,) $\frac{\partial}{\partial x}(\delta u) = \delta\left(\frac{\partial u}{\partial x}\right)$ and $\frac{\partial}{\partial y}(\delta u) = \delta\left(\frac{\partial u}{\partial y}\right)$

<u>Proof:</u>

W. K. T., $\delta u = \epsilon\xi(x, y)$ and $\delta v = \epsilon\eta(x, y)$

$$\frac{\partial}{\partial x}(\delta u) = \frac{\partial}{\partial x}\big(\epsilon\xi(x, y)\big)$$

$$= \epsilon \frac{\partial}{\partial x}\big(\xi(x, y)\big)$$

$$= \epsilon \frac{\partial \xi}{\partial x} = \epsilon\xi_x$$

$$= \delta u_x$$

$$= \delta\left(\frac{\partial u}{\partial x}\right)$$

Now,

$$\frac{\partial}{\partial y}(\delta u) = \frac{\partial}{\partial y}\big(\epsilon\xi(x, y)\big)$$

$$= \epsilon \frac{\partial}{\partial y}\big(\xi(x, y)\big)$$

$$= \epsilon \frac{\partial \xi}{\partial y} = \epsilon \xi_y$$

$$= \delta u_y$$

$$= \delta \left(\frac{\partial u}{\partial y} \right)$$

Lemma-1:

The derivative of the variation with respect to the independent variable is the same as the variation of the derivative

i.e.,

$$\frac{d}{dx}(\delta y) = \delta \left(\frac{dy}{dx} \right)$$

Proof:

$\delta y = \epsilon \eta(x)$ and $\delta y' = \epsilon \eta'(x)$

$$\frac{d}{dx}(\delta y) = \frac{d}{dx}(\epsilon \eta(x))$$

$$= \epsilon \frac{d}{dx}(\eta(x))$$

$$= \epsilon \eta'(x)$$

$$= \delta y'$$

$$= \delta \left(\frac{dy}{dx} \right)$$

Note:

The above lemma is true only when the differentiation is with respect to an independent variable. In general case, it is not true.

Example:

1. Let x and y be two independent variable t.

Now,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{y'}{x'} \; where \; y' = y'^{(t)} = \frac{dy}{dt} \; and \; x' = x'^{(t)} = \frac{dx}{dt}$$

$$\delta\left(\frac{y'}{x'}\right) = \frac{x'\delta y' - y'\delta x'}{x'^2}$$

$$= \frac{\left(\frac{dx}{dt}\right)\delta\left(\frac{dy}{dt}\right) - \left(\frac{dy}{dt}\right)\delta\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$= \frac{\left(\frac{dx}{dt}\right)\left(\frac{d}{dt}\right)\delta y - \left(\frac{dy}{dt}\right)\left(\frac{d}{dt}\right)\delta x}{\left(\frac{dx}{dt}\right)^2}$$

$$= \frac{\left(\frac{d}{dt}\right)\delta y}{\frac{dx}{dt}} - \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}\left(\frac{d}{dt}\right)\delta x$$

$$= \frac{d}{dx}(\delta y) - \frac{dy/dt}{dx/dt}\frac{d/dt(\delta x)}{dx/dt}$$

$$= \frac{d}{dx}(\delta y) - \frac{dy}{dx}\left[\frac{d}{dx}(\delta x)\right]$$

If x is independent, $\delta x = 0$,

$$\delta\left(\frac{y'}{x'}\right) = \frac{d}{dx}(\delta y)$$

If x is dependent, then lemma is not true.

<u>Note:</u>

The quantity δF is called First variation of F. The second variation is defined as the group of second order terms in ϵ in the equation ΔF

<u>Result:</u>

Let F be a functional expressed as the definite integral $I = \int_{x_1}^{x_2} F(x, y, y')dx$, there exists a independent variable.

$$\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx$$

$$= \int_{x_1}^{x_2} \delta F(x, y, y')dx$$

Theorem -1:

For a stationary function, the variation of the integral is zero.

(or)

The integral I is stationary if and only if its variation is vanishes.

i.e., $\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx = 0$ for every permissible variation δy.

Proof:

Let the integral $I = \int_{x_1}^{x_2} F(x, y, y')dx$ be stationary.

To Prove: $\delta I = 0$

I is stationary $\Rightarrow$ Euler's equation satisfied.

i.e., $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

$$\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx$$

$$= \int_{x_1}^{x_2} \delta\, F(x, y, y')dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta\left(\frac{dy}{dx}\right) dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \left[\frac{\partial F}{\partial y'}\,\delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right) \delta y dx$$

$$= 0$$

$$\delta I = 0$$

Assume that $\delta I = 0$

To prove: I is stationary

i.e. T.P: It satisfies the Euler's equation

$$\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx = 0$$

$$\int_{x_1}^{x_2} \delta\, F(x, y, y')dx = 0$$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta\left(\frac{dy}{dx}\right) dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \left[\frac{\partial F}{\partial y'}\ \delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx = 0$$

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right) \delta y dx = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow I \text{ is stationary}$$

Note:

The stationary function for the integral functional is one for which the variation of that integral is zero. Hence for a stationary point of the functional, the differentiable of the function is zero.

Green's Theorem:

If $\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ are defined and continuous on the simple region R bounded by piecewise smooth simple curve C then,

$$\iint_R \frac{\partial \phi}{\partial x} dx\, dy = \oint_C \phi \cos\theta\, ds$$

$$\iint_R \frac{\partial \phi}{\partial y} dx\, dy = \oint_C \phi \sin\theta\, ds$$

Where s is the arclength along the curve C, and θ is the angle between the positive direction of the x −axis and the outward normal drawn at a point on the curve C

Theorem-2:

P. T. the necessary condition for the integral $I = \iint_R F(x, y, u, v, u_x, u_y, v_x. v_y)\, dx\, dy$ to take maximum or minimum value is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial F}{\partial u} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) - \frac{\partial F}{\partial v} = 0$$

(or)

Derive Euler equation for the more general case using Green's theorem.

Proof:

Let us consider the case when the integral to be maximised or minimised is of the form $I = \iint_R F(x, y, u, v, u_x, u_y, v_x. v_y)\, dx\, dy$ ______________(A)

Here x and y are independent variable, u and v are continuously differentiable function of x and y. R is the two dimensional region in the $xy -$ plane.

The condition for I to take maximum or minimum value is $\delta I = 0$

i.e., The condition for stationary is $\delta I = 0$

$$\delta I = \delta \iint_R F(x, y, u, v, u_x, u_y, v_x . v_y)\, dx\, dy = 0$$

$$\Rightarrow \iint_R \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial v} \delta v + \frac{\partial F}{\partial u_x} \delta u_x + \frac{\partial F}{\partial u_y} \delta u_y + \frac{\partial F}{\partial v_x} \delta v_x + \frac{\partial F}{\partial v_y} \delta v_y \right) dx = 0 \_\_\_\_\_\_(1)$$

Here, δu and δv are to continuously differentiable over F and must vanish on the boundary C when $u(x, y)$ and $v(x, y)$ are arbitrary function defined along the curve C

In order to transform the terms involving, the variation of derivatives we make use of the Green's theorem and the formula

$$\iint_R \frac{\partial \phi}{\partial x} dx\, dy = \oint_C \phi \cos\theta\, ds \_\_\_\_\_\_\_\_\_\_(2)$$

$$\iint_R \frac{\partial \phi}{\partial y} dx\, dy = \oint_C \phi \sin\, \theta\, ds \_\_\_\_\_\_\_\_\_\_(3)$$

Where θ is the angle between the positive x –axis and the outward normal at a point of the boundary C of R and s is the arclength along C

Now,

$$\iint_R \frac{\partial F}{\partial u_x} \delta u_x\, dx\, dy = \iint_R \frac{\partial F}{\partial u_x} \delta \left(\frac{\partial u}{\partial x} \right) dx\, dy$$

$$= \iint_R \frac{\partial F}{\partial u_x} \frac{\partial}{\partial x} (\delta u)\, dx\, dy \_\_\_\_\_\_\_(4)$$

Consider

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) = \frac{\partial F}{\partial u_x} \frac{\partial}{\partial x} (\delta u) + \delta u \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right)$$

$$\Rightarrow \frac{\partial F}{\partial u_x} \left(\frac{\partial}{\partial x} \delta u \right) = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \delta u \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right)$$

$$(4) \Rightarrow \iint_R \frac{\partial F}{\partial u_x} \delta u_x\, dx\, dy = \iint_R \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \delta u \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \right] dx\, dy$$

$$= \iint_R \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) dx\, dy - \iint_R \delta u \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) dx\, dy$$

$$= \oint_C \frac{\partial F}{\partial u_x} \delta u \cos\theta \, ds - \iint_R \delta u \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) dx\, dy \_\_\_\_\_\_(5)$$

Now,

$$\iint_R \frac{\partial F}{\partial u_y} \delta u_y \, dx\, dy = \iint_R \frac{\partial F}{\partial u_y} \delta\left(\frac{\partial u}{\partial y}\right) dx\, dy$$

$$= \iint_R \frac{\partial F}{\partial u_y} \frac{\partial}{\partial y} (\delta u) \, dx\, dy$$

Consider,

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y} \delta u\right) = \frac{\partial F}{\partial u_y}\frac{\partial}{\partial y}(\delta u) + \delta u \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)$$

$$\Rightarrow \frac{\partial F}{\partial u_y}\left(\frac{\partial}{\partial y}\delta u\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \delta u \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)$$

Therefore,

$$\iint_R \frac{\partial F}{\partial u_y} \delta u_y \, dx\, dy = \iint_R \left[\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \delta u \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] dx\, dy$$

$$= \iint_R \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) dx\, dy - \iint_R \delta u \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) dx\, dy$$

$$= \oint_C \frac{\partial F}{\partial u_y} \delta u \sin\theta \, ds - \iint_R \delta u \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) dx\, dy \_\_\_\_\_\_\_\_(6)$$

Similarly,

$$\iint_R \frac{\partial F}{\partial v_x} \delta v_x \, dx\, dy = \oint_C \frac{\partial F}{\partial v_x} \delta v \cos\theta \, ds - \iint_R \delta v \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) dx\, dy \_\_\_\_\_\_(7)$$

$$\iint_R \frac{\partial F}{\partial v_y} \delta v_y \, dx\, dy = \oint_C \frac{\partial F}{\partial v_y} \delta v \sin\theta \, ds - \iint_R \delta v \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) dx\, dy \_\_\_\_\_\_(8)$$

$$(1) \Rightarrow \iint_R \frac{\partial F}{\partial u}\delta u \, dxdy + \iint_R \frac{\partial F}{\partial v}\delta v \, dx\, dy + \iint_R \frac{\partial F}{\partial u_x}\delta u_x \, dx\, dy + \iint_R \frac{\partial F}{\partial u_y}\delta u_y dx\, dy$$
$$+ \iint_R \frac{\partial F}{\partial v_x}\delta v_X \, dx\, dy + \iint_R \frac{\partial F}{\partial v_y}\delta v_y \, dx\, dy = 0$$

Substitute equations (5), (6), (7) and (8), we get,

$\iint_R \frac{\partial F}{\partial u}\delta u \, dxdy + \iint_R \frac{\partial F}{\partial v}\delta v \, dx\, dy + \oint_C \frac{\partial F}{\partial u_x}\delta u \cos\theta \, ds - \iint_R \delta u \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) dx\, dy +$
$\oint_C \frac{\partial F}{\partial u_y}\delta u \sin\,\theta \, ds - \iint_R \delta u \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) dx\, dy + \oint_C \frac{\partial F}{\partial v_x}\delta v \cos\theta \, ds -$
$\iint_R \delta v \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) dx\, dy + \oint_C \frac{\partial F}{\partial v_y}\delta v \sin\theta \, ds - \iint_R \delta v \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) dx\, dy = 0$

$$\iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u \, dxdy + \iint_R \left[\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right]\delta v \, dxdy +$$
$$\oint_C \left(\frac{\partial F}{\partial u_x}\cos\theta + \frac{\partial F}{\partial u_y}\sin\theta\right)\delta u \, ds + \oint_C \left(\frac{\partial F}{\partial v_x}\cos\theta + \frac{\partial F}{\partial v_y}\sin\theta\right)\delta v \, ds = 0$$

If the variation δu and δv are independent of each other then the coefficient of δu and δv inside the double integral must each vanish in R giving rise to the Euler's equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0$$

<u>The Natural Boundary Condition:</u>

When u is not prescribed on C,

$$\frac{\partial F}{\partial u_x}\cos\theta + \frac{\partial F}{\partial u_y}\sin\theta = 0$$

When v is not prescribed on C,

$$\frac{\partial F}{\partial v_x}\cos\theta + \frac{\partial F}{\partial v_y}\sin\theta = 0$$

The two Euler's equations are linear and called **Quassi linear equation.**

The two Euler equation represent the necessary equation that the equation $I = \int_{x_1}^{x_2} F(x, y, u, v, u_x, u_y, v_x. v_y)\, dx\, dy$ takes maximum or minimum value subject to the boundary condition along C.

Note:

If the integrant F involves n independent variables and m independent variables together with the partial derivatives of various order with respect to x and y we obtain the Euler equation of m independent variables.

The equation corresponding u is of the form

$$F_u - \left(\frac{\partial}{\partial x} F_{u_x} + \frac{\partial}{\partial y} F_{u_y} + \cdots\right) + \left(\frac{\partial^2}{\partial x^2} F_{u_{xx}} + \frac{\partial^2}{\partial x \partial y} F_{u_{xy}} + \frac{\partial^2}{\partial y^2} F_{u_{yy}} + \cdots\right)$$
$$- \left(\frac{\partial^3}{\partial x^3} F_{u_{xxx}} + \cdots\right) + \left(\frac{\partial^4}{\partial x^4} F_{u_{xxxx}} + \cdots\right) - \cdots = 0$$

Applications:

1. **Obtain the partial differential equation satisfied by the equation of minimal surface.**

Or

Find the surface passing through a simple closed curve C in space having minimum surface area bounded by C

Assume the equation of the surface as $z = z(x, y)$

Then the area to be minimised is given by the integral $\delta = \iint_R \sqrt{1 + z_x^2 + z_y^2} dx\, dy$

where R is the region in the xy −plane bounded by the projection C_0 of C onto the xy −plane, and where z is along C_0

Here,

$$F = \sqrt{1 + z_x^2 + z_y^2} = \left(1 + z_x^2 + z_y^2\right)^{\frac{1}{2}}$$

Euler equation is given by

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0 \_\_\_\_\_\_\_\_\_ (1)$$

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial z_x} = \frac{1}{2}\left(1 + z_x^2 + z_y^2\right)^{\frac{-1}{2}} 2z_x$$

$$= \frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) = \frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right)$$

$$= \frac{\sqrt{1 + z_x^2 + z_y^2}\, z_{xx} - z_x \frac{1}{2}\left(1 + z_x^2 + z_y^2\right)^{-\frac{1}{2}}\left(2z_x z_{xx} + 2z_y z_{yx}\right)}{1 + z_x^2 + z_y^2}$$

$$= \frac{\sqrt{1 + z_x^2 + z_y^2}\, z_{xx} - \left(1 + z_x^2 + z_y^2\right)^{-\frac{1}{2}} z_x\left(z_x z_{xx} + z_y z_{yx}\right)}{1 + z_x^2 + z_y^2}$$

$$= \frac{\left(1 + z_x^2 + z_y^2\right) z_{xx} - z_x^2 z_{xx} - z_x z_y z_{yx}}{\left(1 + z_x^2 + z_y^2\right)^{\frac{3}{2}}}$$

$$= \frac{z_{xx} + z_y^2 z_{xx} - z_x z_y z_{yx}}{\left(1 + z_x^2 + z_y^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right)$$

$$= \frac{\sqrt{1 + z_x^2 + z_y^2}\, z_{yy} - z_y \frac{1}{2}\left(1 + z_x^2 + z_y^2\right)^{-\frac{1}{2}}\left(2z_y z_{yy} + 2z_x z_{xy}\right)}{1 + z_x^2 + z_y^2}$$

$$= \frac{(1 + z_x^2 + z_y^2)z_{yy} - z_y^2 z_{yy} - z_x z_y z_{xy}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}$$

$$= \frac{z_{yy} + z_x^2 z_{yy} - z_x z_y z_{yx}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}$$

$$(1) \Rightarrow \left[\frac{z_{xx} + z_y^2 z_{xx} - z_x z_y z_{yx}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}\right] - \left[\frac{z_{yy} + z_x^2 z_{yy} - z_x z_y z_{yx}}{(1 + z_x^2 + z_y^2)^{\frac{3}{2}}}\right] = 0$$

$$\Rightarrow z_{xx} + z_y^2 z_{xx} - z_x z_y z_{yx} - z_{yy} - z_x^2 z_{yy} - z_x z_y z_{xy} = 0$$

$$\Rightarrow z_{xx}(1 + z_y^2) + z_{yy}(1 + z_x^2) - 2z_x z_y z_{xy} = 0$$

$$\Rightarrow r(1 + q^2) + t(1 + p^2) - 2pqs = 0$$

Where $p = \frac{\partial z}{\partial x} = z_x, q = \frac{\partial z}{\partial y} = z_y, r = \frac{\partial^2 z}{\partial x^2} = z_{xx}, s = \frac{\partial^2 z}{\partial x \partial y} z_{xy}, t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$

Laplace Equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Is called the Laplace equation.

Dirichlet problem:

The problem of determining the function ϕ which satisfy the Laplace equation in the region R is called Dirichlet problem.

Gradient:

The gradient is $\nabla^2 \phi$. i.e..,

$$\nabla^2 \phi = (\phi_x \vec{i} + \phi_y \vec{j} + \phi_z \vec{k})(\phi_x \vec{i} + \phi_y \vec{j} + \phi_z \vec{k})$$

$$= \phi_x^2 + \phi_y^2 + \phi_z^2$$

Note:

The dirichlet problem is equivalent to the variational problem. The solution to the dirichlet problem can be determined by the means of Euler equation of a suitable variational problem and conversely.

1. **Find the function $\boldsymbol{\phi(x,y,z)}$ for which the mean square value of the magnitude of the gradient over the region $\boldsymbol{R}$ of the space is minimum when $\boldsymbol{\phi}$ is given on the boundary of $\boldsymbol{R}$.**

The problem is that we have to find ϕ such that $\delta \iiint_R \left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) dx\, dy\, dz$

$$F = \phi_x^2 + \phi_y^2 + \phi_z^2$$

Euler equation:

$$\frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \phi_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial \phi_z}\right) = 0$$

$$\Rightarrow 0 - \frac{\partial}{\partial x}(2\phi_x) - \frac{\partial}{\partial y}(2\phi_y) - \frac{\partial}{\partial z}(2\phi_z) = 0$$

$$\Rightarrow -2\phi_{xx} - 2\phi_{yy} - 2\phi_{zz} = 0$$

$$\Rightarrow \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\Rightarrow \nabla^2 \phi = 0$$

It satisfy the Laplace equation.

This problem is an example of the dirichlet problem.

Conversely, Let ϕ satisfies the Laplace equation in a region R

To Prove: ϕ takes the equation of the form $\phi_x^2 + \phi_y^2 + \phi_z^2$

Now, ϕ satisfies the Laplace equation.

$$\Rightarrow \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

Multiply both the sides by continuously differentiable function $\nabla\phi$, which vanishes on the boundary of R and integrating over R we get,

$$\iiint_R (\phi_{xx} + \phi_{yy} + \phi_{zz})\delta\phi \, dx\, dy\, dz = 0 \text{ \_\_\_\_\_\_\_\_ } (1)$$

Now,

$$\iiint_R \phi_{xx}\delta\phi \, dx\, dy\, dz = \iint_R \left[\int \phi_{xx}\delta\phi \, dx\right] dy\, dz$$

$$u = \delta\phi\,; \;\; dv = \phi_{xx}dx$$

$$du = d(\delta\phi) \;\;;\;\; v = \phi_x$$

$$du = \delta(d\phi) = \delta\phi_x dx$$

$$= \iint_R \left[[\delta\phi\phi_x]_{x_1}^{x_2} - \int \phi_x\delta\phi_x dx\,\right] dy\, dz$$

$$= -\frac{1}{2}\iiint_R \phi_x 2\delta\phi_x \, dx\, dy\, dz$$

$$= -\frac{1}{2}\iiint_R \delta\phi_x^2 \, dx\, dy\, dz$$

$$= -\frac{\delta}{2}\iiint_R \phi_x^2 \, dx\, dy\, dz$$

Similarly,

$$\iiint_R \phi_{yy}\delta\phi \, dx\, dy\, dz = -\frac{\delta}{2}\iiint_R \phi_y^2 \, dx\, dy\, dz$$

$$\iiint_R \phi_{zz}\delta\phi \, dx\, dy\, dz = -\frac{\delta}{2}\iiint_R \phi_z^2 \, dx\, dy\, dz$$

$$(1) \Rightarrow -\frac{\delta}{2}\iiint_R \phi_x^2 + \phi_y^2 + \phi_z^2 \, dx\, dy\, dz = 0$$

$$\Rightarrow -\delta \iiint_R \phi_x^2 + \phi_y^2 + \phi_z^2 \, dx\, dy\, dz = 0$$

Hence ϕ is such that $\phi = \phi_x^2 + \phi_y^2 + \phi_z^2$ prescribed on the boundary of R

Hence in the dirichlet problem we find a function which satisfies the Laplace equation in a region R and which takes on prescribed values along the boundary of R.

UNIT-2

Constraints and Lagrange multipliers:

1. **Derive the Euler equation along with constraint conditions and Lagrange multipliers.**

Proof:

Let x be an independent variable, u and v be dependent variable such that

$$\delta \int_{x_1}^{x_2} F(x, u, v, u_x, v_x) dx = 0 \quad \_\_\_\_\_\_\_\_\_\_(1)$$

And the constraint is of the form $\phi(u, v) = 0$ _____(2)

Assume that u and v are prescribed at the end points in consistency with equation (2)

Now, equation (1) becomes

$$\int_{x_1}^{x_2} \delta F(x, u, v, u_x, v_x) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial v} \delta v + \frac{\partial F}{\partial u_x} \delta u_x + \frac{\partial F}{\partial v_x} \delta v_x \right) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial v} \delta v \right) + \left(\frac{\partial F}{\partial u_x} \delta \left(\frac{du}{dx} \right) + \frac{\partial F}{\partial v_x} \delta \left(\frac{dv}{dx} \right) \right) \right] dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial v} \delta v \right) + \left(\frac{\partial F}{\partial u_x} \left(\frac{d}{dx} \right) \delta u + \frac{\partial F}{\partial v_x} \left(\frac{d}{dx} \right) \delta v \right) \right] dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v\right) dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial u_x}\left(\frac{d}{dx}\right)\delta u\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial v_x}\left(\frac{d}{dx}\right)\delta v\, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v\right) dx + \left[\frac{\partial F}{\partial u_x}\delta u\right]_{x_1}^{x_2}$$

$$- \int_{x_1}^{x_2} \delta u \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) dx + \left[\frac{\partial F}{\partial v_x}\delta v\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta v \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v\right) dx + 0$$

$$- \int_{x_1}^{x_2} \delta u \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) dx + 0 - \int_{x_1}^{x_2} \delta v \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right)\delta u\, dx + \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\right)\delta v\, dx = 0 \_\_\_\_\_(3)$$

Now,

u and v must satisfy $\phi(u, v) = 0$

Hence the variation δu and δv cannot both be assigned arbitrarily inside (x_1, x_2)

Therefore, the coefficient of δu and δv in equation (3) need not vanish separately.

Now,

$$\phi(u, v) = 0$$

$$\Rightarrow \delta\phi = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\delta u + \frac{\partial \phi}{\partial v}\delta v = 0 \_\_\_\_\_(4)$$

Now,

Multiply equation (4) by λ and taking integration with respect to x in (x_1, x_2) we get,

$$\int_{x_1}^{x_2} \left(\frac{\partial \phi}{\partial u}\lambda\, \delta u + \frac{\partial \phi}{\partial v}\lambda\, \delta v\right) dx = 0 \_\_\_\_\_\_(5)$$

(3) + (5) we get,

$$\Rightarrow \int_{x_1}^{x_2} \left\{\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial \phi}{\partial u}\lambda\right) \delta u\right\} dx$$

$$+ \int_{x_1}^{x_2} \left\{\left(\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) + \frac{\partial \phi}{\partial v}\lambda\right) \delta v\right\} dx = 0 \_\_\_\_\_(6)$$

Equation (6) is true for any value of λ

Let λ be chosen so that the coefficient of δu in equation (6) vanishes, then the single variation δv can be assigned arbitrarily inside (x_1, x_2) and its coefficient also vanish. Thus, we must have,

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial \phi}{\partial u}\lambda = 0$$

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) + \frac{\partial \phi}{\partial v}\lambda = 0$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial F}{\partial u} - \lambda\phi_u = 0 \quad \text{\_\_\_\_\_(7)}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial F}{\partial v} - \lambda\phi_v = 0 \quad \text{\_\_\_\_\_(8)}$$

Multiply (7) by ϕ_v

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\phi_v - \frac{\partial F}{\partial u}\phi_v - \lambda\phi_u\phi_v = 0 \quad \text{\_\_\_\_\_\_(9)}$$

Multiply (8) by ϕ_u

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\phi_u - \frac{\partial F}{\partial v}\phi_u - \lambda\phi_u\phi_v = 0 \quad \text{\_\_\_\_\_\_(10)}$$

$(9) - (10)$ we get,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\phi_v - \frac{\partial F}{\partial u}\phi_v - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\phi_u + \frac{\partial F}{\partial v}\phi_u = 0$$

$$\Rightarrow \left(\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial F}{\partial u}\right)\phi_v - \left(\frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial F}{\partial v}\right)\phi_u = 0 \quad \text{\_\_\_(11)}$$

Equation (2) and (11) gives two conditions in u and v.

Write this method after 4th equation

$$(4) \Rightarrow \frac{\partial \phi}{\partial u}\delta u + \frac{\partial \phi}{\partial v}\delta v = 0$$

$$\phi_u \delta u + \phi_v\, \delta v = 0$$

$$(3) \Rightarrow \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right)\left(\frac{-\phi_v\, \delta v}{\phi_u}\right)dx + \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\right)\delta v\, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{-\frac{\partial F}{\partial u}\phi_v\,\delta v + \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\phi_v\,\delta v + \frac{\partial F}{\partial v}\phi_u\delta v - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\phi_u\delta v}{\phi_u} \right) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left(-\frac{\partial F}{\partial u}\phi_v + \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\phi_v + \frac{\partial F}{\partial v}\phi_u - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\phi_u\right)\delta v\, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{\left(\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial F}{\partial u}\right)\phi_v - \left(\frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial F}{\partial v}\right)\phi_u\right\}\delta v\, dx = 0$$

Equating to zero, the coefficient of δv we get,

$$\left(\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial F}{\partial u}\right)\phi_v - \left(\frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial F}{\partial v}\right)\phi_u = 0$$

Note:

Sometimes the constraints condition may be given in a variation form as

$f.\delta u + g.\delta v = 0$

Instead of $\phi(u, v) = 0$

In this case, replace ϕ_u by f and ϕ_v by g in the equation (7), (8), (11)

Result:

A constraint condition may be expressed by certain definite integral involving the unknown functions or function take on the prescribed value.

1. **Find the Lagrange multiplier λ with the constants of integration arising in the solution of Euler equation, so that the constraint $\int_a^b G\, dx = k$ is satisfied and also the end conditions are satisfied.**

   Proof:

   Suppose that $y(x)$ to be determined such that $I = \int_{x_1}^{x_2} F(x, y, y')dx$___(1)

   minimum or maximum where y is prescribed at the end points,

$$\left.\begin{aligned} y(x_1) = y_1 \\ y(x_2) = y_2 \end{aligned}\right\} \text{\_\_\_\_\_\_}(2)$$

And y also satisfy the single constraint condition $J = \int_{x_1}^{x_2} G(x, y, y')dx = k$ ____(3)

where k is the prescribed constant.

In order to satisfy equation (2) and (3) be define a variate function. We express y in terms of two parameters ϵ_1 and ϵ_2,

$$\delta y(x) = \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x) \text{\_\_\_\_\_\_}(4)$$

Where $\eta_1(x)$ and $\eta_2(x)$ are continuously differentiable function which each vanish at $x = x_1$ and $x = x_2$

$$I(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} F\left(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta'_1 + \epsilon_2\eta'_2\right) dx \text{\_\_\_\_\_}(5)$$

$$J(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} G\left(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta'_1 + \epsilon_2\eta'_2\right) dx = k \text{\_\_\_\_\_}(5)$$

$I(\epsilon_1, \epsilon_2)$ takes a maximum or minimum value subject to the constraint $J(\epsilon_1, \epsilon_2) = k$ when ϵ_1 and ϵ_2

Let introduce the Lagrange multiplier λ, then

$$\frac{\partial I}{\partial \epsilon_1}(\epsilon_1, \epsilon_2) + \lambda \frac{\partial J}{\partial \epsilon_1}(\epsilon_1, \epsilon_2) = 0 \text{\_\_\_\_\_\_\_\_\_\_}(7)$$

$$\frac{\partial I}{\partial \epsilon_2}(\epsilon_1, \epsilon_2) + \lambda \frac{\partial J}{\partial \epsilon_2}(\epsilon_1, \epsilon_2) = 0 \text{\_\_\_\_\_\_\_\_\_\_}(8)$$

When $\epsilon_1 = 0$ and $\epsilon_2 = 0$

$$\frac{\partial I}{\partial \epsilon_1}(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta_1 + \frac{\partial F}{\partial y'}\eta_1'\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta_1 + \frac{\partial F}{\partial y'}\frac{d\eta_1}{dx}\right) dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta_1 \, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d\eta_1}{dx} dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \eta_1 \, dx + \left[\frac{\partial F}{\partial y'} \eta_1\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta_1 \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta_1 \, dx$$

Similarly,

$$\frac{\partial I}{\partial \epsilon_2}(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta_2 \, dx$$

$$\frac{\partial J}{\partial \epsilon_1}(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right) \eta_1 \, dx$$

$$\frac{\partial J}{\partial \epsilon_2}(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right) \eta_2 \, dx$$

$$(7) \Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta_1 \, dx + \lambda \int_{x_1}^{x_2} \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right) \eta_1 \, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right)\right\} \eta_1 \, dx = 0 \quad \text{\_\_\_\_\_\_(9)}$$

Similarly,

$$(8) \Rightarrow \int_{x_1}^{x_2} \left\{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right)\right\} \eta_2 dx = 0 \quad \text{\_\_\_\_\_(10)}$$

If η_2 is chosen such that,

$$\int_{x_1}^{x_2} \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right) \eta_2 \, dx \neq 0$$

Then, the value of λ can be determined from equation (10)

Hence $\eta_1(x)$ is arbitrary.

In equation (9), the coefficient of η_1 is zero.

Hence we desired the Euler's equation,

$$\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)+\lambda\left(\frac{\partial G}{\partial y}-\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right)=0$$

$$\Rightarrow \frac{\partial}{\partial y}(F+\lambda G)-\frac{d}{dx}\left(\frac{\partial}{\partial x}(F+\lambda G)\right)=0$$

The solution of this equation involves 3 constants where constant parameter λ and two constants of integration in correspondence with the 3 condition equation (2) and equation (3) which are to be satisfied in this case.

Note:

1. In order to maximise or minimise an integral $\int_a^b F\,dx$ subject to the constraint $\int_a^b G\,dx = k$. First, write $H = F + \lambda G$ where λ is a constant and maximise or minimise $\int_a^b H\,dx$ subject to no constraint. Hence the Euler equation will become
$$\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right)=0$$
2. If one of the condition $y(x_1) = y_1$ and $y(x_2) = y_2$ is not imposed, the condition $\frac{\partial H}{\partial y\prime} = 0$ must be substituted for it at the end.

1. **To determine the curve of length l which passes through the point $(0,0)$ and $(1,0)$ and for which the area I between the curve and the x −axis is maximum. We have to maximise the integral $I = \int_0^1 y\,dx$ subject to the end condition $y(0) = y(1) = 0$ and to the constraint $J = \int_0^1 (1+y'^2)^{\frac{1}{2}}\,dx = l$ where l is a constant greater than unity.**

$$k = I + \lambda J$$
$$= \int_0^1 y\,dx + \lambda\int_0^1 (1+y'^2)^{\frac{1}{2}}\,dx$$
$$= \int_0^1 \left[y + \lambda(1+y'^2)^{\frac{1}{2}}\right] dx$$

Here,

$$H = y + \lambda(1 + y'^2)^{\frac{1}{2}}$$

Euler equation:

$$\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) - \frac{\partial H}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dx}\left[0 + \frac{\lambda}{2}(1 + y'^2)^{\frac{-1}{2}}(2y')\right] - 1 = 0$$

$$\Rightarrow \frac{d}{dx}\left[\frac{\lambda y'}{(1 + y'^2)^{\frac{1}{2}}}\right] - 1 = 0$$

$$\Rightarrow \frac{d}{dx}\left[\frac{\lambda y'}{(1 + y'^2)^{\frac{1}{2}}}\right] = 1$$

Integrating on both sides,

$$\frac{\lambda y'}{(1 + y'^2)^{\frac{1}{2}}} = x + c$$

$$\Rightarrow x = \frac{\lambda y'}{(1 + y'^2)^{\frac{1}{2}}} - c$$

$$\Rightarrow x = \frac{\lambda y'}{(1 + y'^2)^{\frac{1}{2}}} + c_1$$

$$\Rightarrow x - c_1 = \frac{\lambda y'}{(1 + y'^2)^{\frac{1}{2}}}$$

Squaring on both sides

$$\Rightarrow (x - c_1)^2 = \frac{\lambda^2 y'^2}{(1 + y'^2)}$$

$$\Rightarrow (x - c_1)^2(1 + y'^2) = \lambda^2 y'^2$$

$$\Rightarrow (x - c_1)^2 + (x - c_1)^2 y'^2 = \lambda^2 y'^2$$

$$\Rightarrow (x - c_1)^2 + y'^2[(x - c_1)^2 - \lambda^2] = 0$$

$$\Rightarrow y'^2 = \frac{-(x-c_1)^2}{(x-c_1)^2-\lambda^2}$$

$$\Rightarrow y'^2 = \frac{(x-c_1)^2}{\lambda^2-(x-c_1)^2}$$

$$\Rightarrow y' = \frac{x-c_1}{\sqrt{\lambda^2-(x-c_1)^2}}$$

$$\frac{dy}{dx} = \pm\frac{x-c_1}{\sqrt{\lambda^2-(x-c_1)^2}}$$

$$dy = \pm\frac{(x-c_1)dx}{\sqrt{\lambda^2-(x-c_1)^2}}$$

Integrating on both sides,

$$\int dy = \pm\int\frac{(x-c_1)dx}{\sqrt{\lambda^2-(x-c_1)^2}}$$

Let $t = \lambda^2-(x-c_1)^2$

$dt = -2(x-c_1)dx$

$-\frac{dt}{2} = (x-c_1)dx$

$$y = \pm\int\frac{dt/2}{\sqrt{t}} + c_2$$

$$= \frac{1}{2}\int\frac{dt}{\sqrt{t}} + c_2$$

$$= \frac{1}{2}\left[\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c_2$$

$$= \frac{1}{2}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right] + c_2$$

$$= (\lambda^2-(x-c_1)^2)^{\frac{1}{2}} + c_2$$

Therefore,

$$(y - c_2) = (\lambda^2 - (x - c_1)^2)^{\frac{1}{2}}$$

Squaring on both sides,

$$(y - c_2)^2 = (\lambda^2 - (x - c_1)^2)$$

$$\Rightarrow (x - c_1)^2 + (y - c_2)^2 = \lambda^2$$

The three constraints are to be determined so that the circle passes through the end points and the condition that the length of the arc is λ.

<u>Parametric representation of the above equation:</u>

Let t be the parameter such that $x = x(t)$ and $y = y(t)$. Then,

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})dt = maximum$$

$$J = \int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\, dt = l$$

Where,

$$x(t_1) = 0 \; and \; x(t_2) = 1$$

$$y(t_1) = 0 \; and \; y(t_2) = 1$$

Here the . denote the differentiation with respect to t

Let $k = I + \lambda J$

$$= \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})dt + \lambda \int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\, dt$$

$$= \int_{t_1}^{t_2} \left[\frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\right] dt$$

Here $H = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}$

Euler equation:

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) - \frac{\partial H}{\partial x} = 0 \_\_\_\_\_\_(1)$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) - \frac{\partial H}{\partial y} = 0 \_\_\_\_\_\_(2)$$

$$\frac{\partial H}{\partial \dot{x}} = \frac{1}{2}(-y) + \lambda \frac{1}{2}(\dot{x}^2 + \dot{y}^2)^{\frac{-1}{2}}\, 2\dot{x}$$

$$= -\frac{y}{2} + \lambda \dot{x}\, (\dot{x}^2 + \dot{y}^2)^{\frac{-1}{2}}$$

$$\frac{\partial H}{\partial x} = \frac{1}{2}\dot{y}$$

$$\frac{\partial H}{\partial \dot{y}} = \frac{1}{2}x + \lambda \frac{1}{2}(\dot{x}^2 + \dot{y}^2)^{\frac{-1}{2}}\, 2\dot{y}$$

$$= -\frac{x}{2} + \lambda \dot{y}\, (\dot{x}^2 + \dot{y}^2)^{\frac{-1}{2}}$$

$$\frac{\partial H}{\partial y} = \frac{1}{2}\dot{x}$$

$$(1) \Rightarrow \frac{d}{dt}\left[-\frac{y}{2} + \lambda \dot{x}\, (\dot{x}^2 + \dot{y}^2)^{\frac{-1}{2}}\right] - \frac{1}{2}\dot{y} = 0$$

$$\frac{d}{dt}\left[-\frac{y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = \frac{1}{2}\dot{y} \_\_\_\_\_(3)$$

$$(2) \Rightarrow \frac{d}{dt}\left[-\frac{x}{2} + \lambda \dot{y}\, (\dot{x}^2 + \dot{y}^2)^{\frac{-1}{2}}\right] - \frac{1}{2}\dot{x} = 0$$

$$\frac{d}{dt}\left[-\frac{x}{2} + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = \frac{1}{2}\dot{x} \_\_\_\_\_\_(4)$$

Solving (3) and (4) we get the equation of the circle $(x - c_1)^2 + (y - c_2)^2 = \lambda^2$

This method is use if $l > \frac{\pi}{2}$

If $l > \frac{\pi}{2}$ then y is not a single valued function of x so we use the parametric representation with parameter t

Variable End Points:

1. **Derive Euler equation with variable end points or derive Transveraslity condition.**

Proof:

In some variational problem the boundary of the region of integration is not completely specified but is to be determined together with the unknown function.

Let $y(x)$ is to be determined such that

$$\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx = 0 \text{ \_\_\_\_\_}(1)$$

Where x_1 is fixed and the value of $y(x_1)$ is given as $y(x_1) = y_1$ ______(2)

But the point $\big(x_2, y(x_2)\big)$ is required to lie on a certain curve $y = g(x)$ so that $y(x_2) = g(x_2)$_____(3) where $g(x)$ is the given function of x but x_2 is not preassigned.

Here x_2 my be vary

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial x}\delta x\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y'\, dx = 0$$

$$\Rightarrow [F\delta x]_{x_2}^{x_1} + \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta\left(\frac{dy}{dx}\right) dx = 0$$

$$\Rightarrow \left[F_{x=x_2}\delta x_2\right] + \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)\, dx = 0$$

$$\Rightarrow \left[F_{x=x_2}\delta x_2\right] + \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\delta y\, dx + \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_2}^{x_1} - \int_{x_1}^{x_2} \delta y \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx = 0$$

$$\Rightarrow \left[F_{x=x_2}\delta x_2\right] + \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_2}^{x_1} + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx = 0 \text{ \_\_\_\_}(4)$$

In order to relate δx_2 to $\delta y\ (x_2)$ we must use the fact that the variate end points must remain on the curve $y = g(x)$

Thus, if the true function $y(x)$ is changed to

$$y(x) + \Delta y(x) = y(x) + \delta y(x)$$
$$= y(x) + \epsilon\eta(x)$$

And if x_2 correspondingly changed to Δx_2 then the requirement that the new end point lie on $y = g(x)$ is of the form

$$y(x_2 + \Delta x_2) + \epsilon\eta(x_2 + \Delta x_2) = g(x_2 + \Delta x_2) \_\_\_\_\_\_ (5)$$

$$(5) - (3) \Rightarrow y(x_2 + \Delta x_2) + \epsilon\eta(x_2 + \Delta x_2) - y(x_2) = g(x_2 + \Delta x_2) - g(x_2)$$

$$\Rightarrow \; y(x_2 + \Delta x_2) - y(x_2) + \epsilon\eta(x_2 + \Delta x_2) = g(x_2 + \Delta x_2) - g(x_2)$$

$$\Rightarrow y'(x_2)\Delta x_2 + \epsilon\eta(x_2 + \Delta x_2) = g'(x_2) + higher\ order\ in\ \epsilon$$

$$\Rightarrow [y'(x_2) - g'(x_2)]\Delta x_2 = -\epsilon\eta(x_2 + \Delta x_2) + higher\ order\ in\ \epsilon$$

$$\Rightarrow \epsilon\eta(x_2 + \Delta x_2) = [g'(x_2) - y'(x_2)]\Delta x_2 + higher\ order\ in\ \epsilon$$

$$\Rightarrow \epsilon\eta(x_2) = [g'(x_2) - y'(x_2)]\Delta x_2 + higher\ order\ in\ \epsilon$$

$$\Rightarrow \Delta x_2 = \frac{\epsilon\eta(x_2)}{g'(x_2) - y'(x_2)} + higher\ order\ in\ \epsilon$$

$$\Rightarrow \delta x_2 = \frac{\epsilon\eta(x_2)}{g'(x_2) - y'(x_2)}$$

$$\Rightarrow \delta x_2 = \frac{\delta y(x_2)}{g'(x_2) - y'(x_2)}$$

$$(4) \Rightarrow F_{x=x_2}\left(\frac{\delta y(x_2)}{g'(x_2) - y'(x_2)}\right) + \left[\frac{\partial F}{\partial y'}\delta y\right]_{x_2}^{x_1} + \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx = 0$$

$$\Rightarrow \left[\frac{F}{g' - y'} + \frac{\partial F}{\partial y'}\right]_{x=x_2} \delta y(x_2) + \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx = 0$$

Which gives the Euler equation subject to the condition (2) $\Rightarrow y(x_1) = y_1$ and to the condition

$$\left[\frac{F}{g' - y'} + \frac{\partial F}{\partial y'}\right]_{x=x_2} = 0$$

$$\Rightarrow F + (g' - y')\frac{\partial F}{\partial y'} = 0$$

This condition is called the Transversality condition.

Example:

1. **Let $F = (1+y'^2)^{\frac{1}{2}}$ and $g(x) = mx + b$ where m and b are prescribed constant. Find the transversality condition.**

   The Transversality condition is given by

$$F + (g' - y')\frac{\partial F}{\partial y'} = 0 \_\_\_\_\_(1)$$

Now,

$$\frac{\partial F}{\partial y'} = \frac{1}{2}(1+y'^2)^{\frac{-1}{2}}2y'$$

$$= \frac{y'}{\sqrt{1+y'^2}}$$

$$g'(x) = m$$

$$(1) \Rightarrow \sqrt{1+y'^2} + [m - y']\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{1+y'^2+my'-y'^2}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow 1 + my' = 0$$

$$\Rightarrow my' = 0$$

$$\Rightarrow my' = -1$$

$$\Rightarrow m\left(\frac{dy}{dx}\right) = -1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{m} \;\; at \; x = x_2$$

$$\Rightarrow dy = -\frac{1}{m}dx \; at \; x = x_2$$

Integrating,

$$y = -\frac{1}{m}x + c$$

Which is the straight line perpendicular to the given line $g(x) = mx + b$

Therefore, the shortest distance from the point (x_1, y_1) to the nearest point on the line $y = mx + b$ is measured in the direction perpendicular to that line.

Strum Lioville's Problem

Consider the determination of the stationary value of the quantity λ defined by

$$\lambda = \frac{\int_a^b (py'^2 - qy^2)dx}{\int_a^b ry^2\,dx}$$

Where p, q, r are given functions of independent variable x

Proof:

Let

$$\lambda = \frac{\int_a^b (py'^2 - qy^2)dx}{\int_a^b ry^2\,dx} = \frac{I_1}{I_2}$$

The variation of λ is of the form

$$\delta\lambda = \delta\left(\frac{I_1}{I_2}\right)$$

$$= \frac{I_2\delta I_1 - I_1\delta I_2}{I_2^2}$$

$$= \frac{\delta I_1}{I_2} - \frac{I_1}{I_2}\frac{\delta I_2}{I_2}$$

$$= \frac{\delta I_1}{I_2} - \lambda\frac{\delta I_2}{I_2}$$

$$= \frac{1}{I_2}[\delta I_1 - \delta I_2] \_\_\_\_\_\_(1)$$

Now,

$$\delta I_1 = \delta\int_a^b (py'^2 - qy^2)dx$$

$$= \int_a^b [\delta(py'^2) - \delta(qy^2)]dx$$

$$= \int_a^b p\, 2y'\, \delta y' dx - \int_a^b q\, 2\, y\, \delta y\, dx$$

$$= 2\left[\int_a^b py'\delta y' dx - \int_a^b q\, y\, \delta y\, dx\right]$$

$$= 2\left[\int_a^b py'\delta\left(\frac{dy}{dx}\right) dx - \int_a^b q\, y\, \delta y\, dx\right]$$

$$= 2\left[\int_a^b py'\frac{d}{dx}(\delta y) dx - \int_a^b q\, y\, \delta y\, dx\right]$$

$$= 2\left[[py'\delta y]_a^b - \int_a^b \delta y \frac{d}{dx}(py') dx - \int_a^b q\, y\, \delta y\, dx\right]$$

$$= 2\left[[py'\delta y]_a^b - \int_a^b \delta y (py')' dx - \int_a^b q\, y\, \delta y\, dx\right]$$

$$= [2py'\delta y]_a^b - 2\left[\int_a^b [(py')' + q\, y]\, \delta y\, dx\right] \_\_\_\_\_\_ (2)$$

$$\delta I_2 = \delta \int_a^b r y^2 dx$$

$$= \int_a^b \delta(ry^2) dx = \int_a^b r2y\delta y\, dx \_\_\_\_\_\_ (3)$$

$$(1) \Rightarrow \delta\lambda = \frac{1}{I_2}\left[[2py'\delta y]_a^b - 2\left[\int_a^b [(py')' + q\, y]\, \delta y\, dx\right] - \lambda \int_a^b r2y\delta y\, dx\right]$$

$$\delta\lambda = \frac{1}{\int_a^b ry^2\, dx}\left[[2py'\delta y]_a^b - 2\left[\int_a^b [(py')' + q\, y + \lambda ry]\, \delta y\, dx\right]\right]$$

For stationary values of λ we have $\delta\lambda = 0$

$$\Rightarrow \frac{1}{\int_a^b ry^2\, dx}\left[[2py'\delta y]_a^b - 2\left[\int_a^b [(py')' + q\, y + \lambda ry]\, \delta y\, dx\right]\right] = 0$$

$$\Rightarrow [2py'\delta y]_a^b - 2\left[\int_a^b [(py')' + q\, y + \lambda ry]\, \delta y\, dx\right] = 0$$

Thus, the condition $\delta\lambda = 0$ leads to the Euler equation in the form

$$(py')' + qy + \lambda ry = 0$$

$$i.e.,, \frac{d}{dx}(py') + qy + \lambda ry = 0$$

If we apply the natural boundary condition that $py' = 0$ at the end points where y is not preassigned $y(a)$ is prescribed or $(py')_{x=a} = 0$

In particular, when the boundary conditions are homogenous of the form $y(a) = 0\ or$ $y'(a) = 0, y(b) = 0\ or\ y'(b) = 0$, the problem is one of the general Strum Lioville's Problem.

1. **P. T the stationary value of λ must be the characteristic number of the problem.**

To verify this let λ_k and $\phi_k(x)$ be corresponding characteristic quantity so that

$$(p\phi_k')' + q\phi_k + \lambda_k r\phi_k = 0 \text{ \_\_\_\_\_\_\_\_(1)}$$

$$\lambda = \frac{\int_a^b (py'^2 - qy^2)dx}{\int_a^b ry^2\,dx}$$

$$= \frac{\int_a^b py'y'dx - \int_a^b qy^2dx}{\int_a^b ry^2\,dx}$$

$$= \frac{\int_a^b py'\frac{dy}{dx}dx - \int_a^b qy^2dx}{\int_a^b ry^2\,dx}$$

$$= \frac{[py'y]_a^b - \int_a^b y\frac{d}{dx}(py')\,dx - \int_a^b qy^2dx}{\int_a^b ry^2\,dx}$$

$$= \frac{0 - \int_a^b y(py')'\,dx - \int_a^b qy^2dx}{\int_a^b ry^2\,dx}$$

$$= \frac{-\int_a^b (y(py')' + qy^2)dx}{\int_a^b ry^2\,dx}$$

$$= \frac{-\int_a^b ((py')' + qy)y dx}{\int_a^b ry^2\, dx}$$

Replace y by ϕ_k

$$\lambda = \frac{-\int_a^b ((p\phi_k{}')' + q\phi_k)\phi_k dx}{\int_a^b r\phi_k^2\, dx}$$

$$= \frac{-\int_a^b (-\lambda_k r\phi_k)\phi_k dx}{\int_a^b r\phi_k^2\, dx}$$

$$= \frac{\int_a^b (\lambda_k r) dx}{\int_a^b r\, dx}$$

$$= \frac{\lambda_k \int_a^b r dx}{\int_a^b r\, dx}$$

Therefore $\lambda = \lambda_k$ when $y = \phi_k$

Note:

1. Arbitrarily the constraint $\int_a^b ry^2 dx = 1$ in homogenous case then

$$\lambda = \int_a^b (py'^2 - qy^2) dx$$

   Hence the stationary condition takes the form

$$\delta\lambda = \delta \int_a^b (py'^2 - qy^2) dx = 0$$

   Where y must satisfy $\int_a^b ry^2 dx = 1$

   Also, $\delta\lambda = \delta(I_1 - \lambda I_2)$

   In this form, the constant λ play the role of the Lagrange multiplier and is to be determine together with the function y so that $I_1 - \lambda I_2$ is stationary and $y(x) \neq 0$

   This condition for which $\int_a^b ry^2 dx = 1$ is called the **normalising condition**.

2. If the condition $\int_a^b ry^2 dx = 1$ is supressed then the condition

$$\delta\lambda = \delta \int_a^b (py'^2 - qy^2)dx = 0$$

Determine only one stationary function $y = 0$

If this condition $\int_a^b ry^2 dx$ is added the problem has a infinite set of stationary function for each of which λ is stationary for small variations in y

<u>Hamilton's Principle (Basic principle of mathematical physics):</u>

1. **Derive the general form the Hamilton principle and hence derive the Hamilton principle if the force field is conservative.**

<u>Proof:</u>

Consider a single particle P of mass m moving subject to a force field.

Let O be the fixed origin if the vector from a fixed origin to a particle P at a time t is denoted by $\vec{r}$, then by Newton's law of motion,

The path of the particle is given by the vector equation

i.e., $\vec{f} = m\vec{a}$, where $\vec{a}$ is the variations in the acceleration

$$\vec{f} = m\frac{d^2\vec{r}}{dt^2}$$

$$\Rightarrow m\frac{d^2\vec{r}}{dt^2} - \vec{f} = 0 \_\_\_\_(1)$$

Where $\vec{f}$ is the force acting on the particles.

Consider any other path $\vec{r} + \delta\vec{r}$

Let the true path and the variate path coincide at two distinct instants $t = t_1$ and $t = t_2$

i.e., The variation $\delta\vec{r}$ at those two instants is zero.

i.e., $[\delta\vec{r}]_{t_1} = 0, [\delta\vec{r}]_{t_2} = 0 \backslash$

Consider the equation (1)

$$(1) \Rightarrow m\frac{d^2\vec{r}}{dt^2} - \vec{f} = 0$$

Taking dot product with $\delta\vec{r}$ we get

$$\Rightarrow m\frac{d^2\vec{r}}{dt^2}.\delta\vec{r} - \overrightarrow{f}.\delta\vec{r} = 0$$

Integrating,

$$\int_{t_1}^{t_2} m\frac{d^2\vec{r}}{dt^2}.\delta\vec{r}\ dt - \int_{t_1}^{t_2}\vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow m\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right).\delta\vec{r}\ dt - \int_{t_1}^{t_2}\vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow m\left[\left[\delta\vec{r}\frac{d\vec{r}}{dt}\right]_{t_2}^{t_1} - \int_{t_1}^{t_2}\frac{d\vec{r}}{dt}\frac{d}{dt}(\delta\vec{r})\ dt\right] - \int_{t_1}^{t_2}\vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow 0 - m\int_{t_1}^{t_2}\frac{d\vec{r}}{dt}\frac{d}{dt}(\delta\vec{r})\ dt - \int_{t_1}^{t_2}\vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow -\int_{t_1}^{t_2} m\,\frac{d\vec{r}}{dt}\delta\left(\frac{d\vec{r}}{dt}\right)\ dt - \int_{t_1}^{t_2}\vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow -\int_{t_1}^{t_2}\frac{1}{2}m\,\delta\left(\frac{d\vec{r}}{dt}\right)^2\ dt - \int_{t_1}^{t_2}\vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow -\left[\int_{t_1}^{t_2}\frac{1}{2}m\,\delta\left(\frac{d\vec{r}}{dt}\right)^2 + \vec{f}.\delta\vec{r}\,dt\right] = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\delta\left(\frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2\right) + \vec{f}.\delta\vec{r}\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\delta T + \vec{f}.\delta\vec{r}\,dt = 0\ \ where\ T = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2$$

This is the require Hamilton principle in its general form as applied to the motion of the single particle.

If the Force field is conservative then, the above Hamilton principle states that the motion is such that the integral of the difference between kinetic energy and potential energy is stationary for true path.

Let the force field $\vec{f}$ be conservative. Let X, Y, Z be co-ordinate axis.

$$\vec{f} = X\vec{\imath} + Y\vec{\jmath} + Z\vec{k}$$

And

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{f}.d\vec{r} = Xdx + Ydy + Zdz$$

$\vec{f}$ is conservative if and only if $\vec{f}.d\vec{r}$ is a differential $\delta\phi$ of a single valued function ϕ

This function ϕ is called **force potential** and its negative is called the **potential energy N** (say) .

Here, ϕ and V involved an irrelevant additive constants.

$$\delta\vec{r} = \delta x\vec{i} + \delta y\vec{j} + \delta z\vec{k}$$

$$\vec{f}.\delta\vec{r} = X\delta x + Y\delta y + Z\delta z = \delta\phi$$

Where $\phi = \phi(x, y, z)$

$$X\delta x + Y\delta y + Z\delta z = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$$

Where $X = \frac{\partial\phi}{\partial x}, Y = \frac{\partial\phi}{\partial y}, Z = \frac{\partial\phi}{\partial z}$

The Hamilton's principle for general case

$$\int_{t_1}^{t_2} \delta T + \vec{f}.\delta\vec{r}\, dt = 0$$

Here, the force field is conservative

$$\vec{f}.\delta\vec{r} = \delta\phi$$

$$\int_{t_1}^{t_2} (\delta T + \delta\phi)dt = 0$$

$$\int_{t_1}^{t_2} \delta(T + \phi)dt = 0$$

The negative of the force potential ϕ is the potential energy V.

$$\delta\int_{t_1}^{t_2} (T - V)dt = 0$$

Hence when the force acting are conservative, the Hamilton principle takes the form

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0$$

Note:

We can show that this integral is minimum compare with that corresponding to any neighbouring to any neighbouring path having the same minimal configuration. Nature tends to equalize the kinetic energy and potential energy over motion.

Lagrangian function:

The energy differential $L = T - V$ is called Kinetic potential or Lagrangian function.

Hence the Hamilton principle becomes

$$\delta \int_{t_1}^{t_2} L. dt = 0$$

Note:

If non-conservative force field are present the potential energy does not exist and we must have

$$\int_{t_1}^{t_2} \vec{f}. \delta\vec{r} \, dt = 0$$

Hence we get,

$$\delta \int_{t_1}^{t_2} T \, dt = 0$$

Definition:

$\vec{f}. \delta\vec{r}$ is a element of work done by the force $\vec{f}$ in the small displacement $\delta\vec{r}$.

If the force $\vec{f}$ is conservative, the element of work done $\vec{f}. \delta\vec{r} = \delta\phi = -\delta V$

Note:

If we have the system of n particles the above derivation can be given by summation and for continuous system it is given by integration.

Result:

Consider the k^{th} particle of mass m_k with position vector $\overrightarrow{r_k}$ and subject to the force $\overrightarrow{f_k}$ then the total kinetic energy is given by

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k \left(\frac{d\overrightarrow{r_k}}{dx}\right)^2$$

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k V^2$$

And the total work done by the force is given by

$$\sum_{k=1}^{N} \overrightarrow{f_k}\delta\overrightarrow{r_k}$$

Conservative force:

If the component of the force can be derived from the scalar valued function ϕ then we say that $\vec{f}$ is conservative.

i.e., If there exists a function ϕ such that $\phi = \phi(x, y, z)$ and $\vec{f}.\delta\vec{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$

Then, $\vec{f}$ is conservative.

Generalised Co-ordinates:

Consider a dynamatical system of n degree of freedoms it is possible to choose n independent geometrically quantities which uniquely specifies all components of the system. These geometrical co-ordinates are called generalised co-ordinates.

1. **Derive the Lagrange's equation for conservative system having n generalized co-ordinates.**

   Proof:

   Let the n generalised co-ordinates be $q_1, q_2, \ldots, q_n$.

The total kinetic energy T may depend upon the generalized co-ordinates $q_1, q_2, \dots, q_n$ also the rate of change with respect to time.

i.e., the generalised velocities is given by $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$. For conservative system, the total potential energy V is a function of only position and hence it does not depend upon the generalized velocities.

Let q_i' are given small displacement δq_i.

Now,

$V = V(q_1, q_2, \dots, q_n)$

$$\Rightarrow \delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \dots + \frac{\partial V}{\partial q_n}\delta q_n$$

The work done by the force system is given by

$$-\delta V = \delta Q$$

$$= Q_1\delta q_1 + Q_2\delta q_2 + \dots + Q_n\delta q_n$$

Where $Q_1 = -\frac{\partial V}{\partial q_1}, Q_2 = -\frac{\partial V}{\partial q_2}, \dots, Q_n = -\frac{\partial V}{\partial q_n}$

The quantity $Q_i\delta q_i$ is the work done by the force in a small displacement δq_i. The force Q may or may not have the dimension of true force and hence they are called a generalised force.

If q_i is linear displacement then, Q_i is truly a force.

If q_i is angular displacement then Q_i is torque.

Consider the conservative system the kinetic energy T, is in terms of nq and $n\dot{q}$ but, the total potential energy V is expressed in terms of q's alone.

The Hamilton principle for conservative system is given by

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0$$

The Associated Euler equation with respect to n generalised co-ordinates is given by $q_1, q_2, \dots, q_n$ is

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial\dot{q_k}}\right)-\frac{\partial(T-V)}{\partial q_k}=0$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q_k}}\right)-\frac{d}{dt}\left(\frac{\partial V}{\partial\dot{q_k}}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}=0$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q_k}}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}=0$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q_k}}\right)-\frac{\partial T}{\partial q_k}=-\frac{\partial V}{\partial q_k}$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q_k}}\right)-\frac{\partial T}{\partial q_k}=Q_k, where\ Q_k=-\frac{\partial V}{\partial q_k}$$

This is Lagrange's equation for a conservative system having n generalised co-ordinates and there are n equations for each $q_i's$.

Degrees of Freedom:

The number of direction in which the particle can move freely.

Or

The number of independent co-ordinates required to specify completely the position and orientation of the particle in space is called degree of freedom.

1. **Derive Lagrange equation of Simple pendulum.**

Proof:

Consider a simple pendulum of point mass m suspended by an inextensible string of length l.

The position of the mass m is completely determined by the angle θ between the deflector and the equilibrium position of the string.

Here the degree of freedom is 1 and we have only one generalised co-ordinate θ.

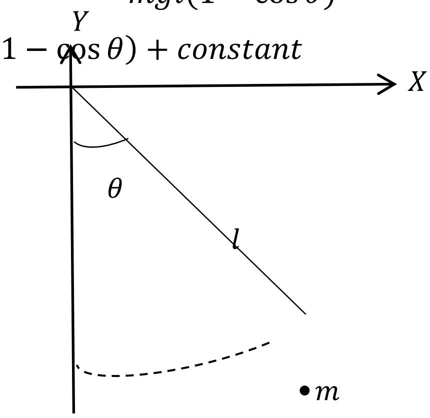
Here,

$$T=\frac{1}{2}m\left(l\dot{\theta}\right)^2=\frac{1}{2}ml^2\dot{\theta}^2$$

The work done by gravity in lifting the mass from its equilibrium position to the position θ is given by

$$workdone = -mgl - (-mgl\cos\theta)$$
$$= -mgl + mgl\cos\theta$$
$$= -mgl(1-\cos\theta)$$

Potential energy $V = mgl(1-\cos\theta) + constant$



The Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} ml^2 2\,\dot{\theta} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - 0 + mgl(\sin\theta) = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\Rightarrow ml\left(l\ddot{\theta} + g\sin\theta\right) = 0$$

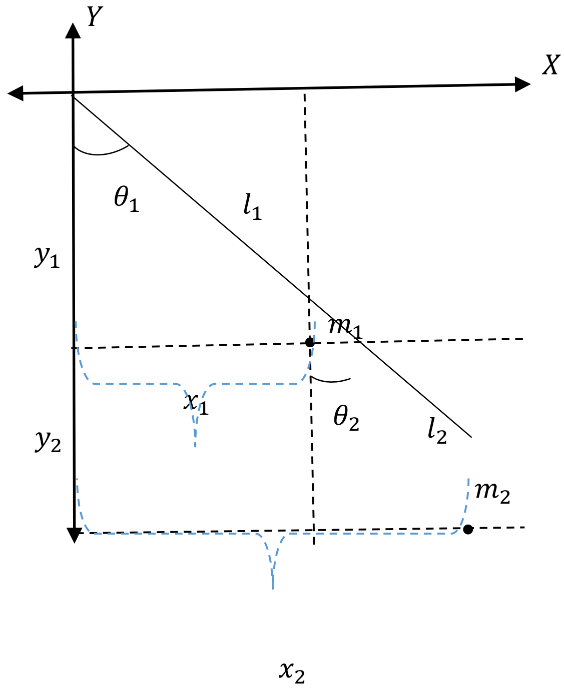
$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$$

This is the required Lagrange equation for the motion of the simple pendulum.

2. Derive Lagrange equation for compound pendulum

The system can be completely specified if we know the angle. The compound pendulum has two degrees of freedom. Hence there are two generalised co-ordinated θ_1 and θ_2.

Let the co-ordinate Y of m_1 and m_2 be (x_1, y_1) and (x_2, y_2)



$$x_1 = l_1 \sin\theta_1$$

$$y_1 = -l_1 \cos\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

Kinetic energy

$$T = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \_\_\_\_\_\_\_\_ (1)$$

$$\dot{x}_1 = l_1 \cos\theta_1 \, \dot{\theta}_1$$

$$\dot{x}_2 = l_1 \cos\theta_1 \, \dot{\theta}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2$$

$$\dot{y}_1 = l_1 \sin \theta_1 \, \dot{\theta}_1$$

$$\dot{y}_2 = l_1 \sin\theta_1 \, \dot{\theta}_1 + l_2 \, \sin\theta_2 \, \dot{\theta}_2$$

$$(1) \Rightarrow T = \frac{1}{2} m_1 \left(l_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2 + l_1^2 \sin^2 \, \theta_1 \, \dot{\theta}_1^2 \right) + \frac{1}{2} m_2 (l_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2 + l_2^2 \, \cos^2\theta_2 \, \dot{\theta}_2^2$$

$$+ 2 l_1 l_2 \, \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + \; l_1^2 \sin^2 \, \theta_1 \, \dot{\theta}_1^2 + l_2^2 \, \sin^2\theta_2 \, \dot{\theta}_2^2$$

$$+ 2 l_1 l_2 \, \dot{\theta}_1 \dot{\theta}_2 \sin \, \theta_1 \sin \, \theta_2)$$

$$= \frac{1}{2} m_1 \left(l_1^2 \dot{\theta}_1^2 \, (\cos^2\theta_1 + \sin^2 \, \theta_1 \,) \right)$$

$$+ \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 \, (\cos^2\theta_1 + \sin^2 \, \theta_1 \,) + l_2^2 \dot{\theta}_2^2 \, (\cos^2\theta_2 + \sin^2 \, \theta_2 \,) \right.$$

$$\left. + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin \, \theta_1 \sin \, \theta_2 \,) \right)$$

$$= \frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \_\_\_\_\_\_ (2)$$

Potential energy $V = m_1 g y_1 + m_2 g y_2 + constant$

$$= m_1 g \, (-l_1 \cos\theta_1) + m_2 g (-l_1 \cos\theta_1 - l_2 \cos\theta_2) + constant$$

$$= g l_1 \cos\theta_1 (-m_1 - m_2) - m_2 g l_2 \cos\theta_2 + constant$$

$$= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 l_2 g \cos\theta_2 + constant \_\_\_\_\_\_\_\_\_\_\_\_ (3)$$

The Lagrange equation is

$$\left. \begin{aligned} \frac{d}{dT}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0 \\ \frac{d}{dT}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0 \end{aligned} \right\} \_\_\_\_\_\_\_\_\_\_ (I)$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = \frac{1}{2} l_1^2 \, 2 \, \dot{\theta}_1 (m_1 + m_2) + m_2 \left(l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

$$= l_1^2 \dot{\theta}_1 (m_1 + m_2) + m_2 \left(l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

$$\frac{\partial T}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = \frac{1}{2} m_2 l_2^2 \, 2 \dot{\theta}_2 + m_2 \left(l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \right)$$

$$= m_2 l_2^2 \dot{\theta}_2 + m_2\left(l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right)$$

$$\frac{\partial T}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\dot{\theta}_2(-\sin(\theta_1 - \theta_2))\left(\dot{\theta}_1 - \dot{\theta}_2\right) + \cos(\theta_1 - \theta_2)\ddot{\theta}_2\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \left[\dot{\theta}_1(-\sin(\theta_1 - \theta_2))\left(\dot{\theta}_1 - \dot{\theta}_2\right) + \cos(\theta_1 - \theta_2)\ddot{\theta}_1\right]$$

$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 l_2 g \cos\theta_2 + constant$$

$$\frac{\partial V}{\partial \theta_1} = (m_1 + m_2) g l_1 \sin\theta_1$$

$$\frac{\partial V}{\partial \theta_2} = m_2 l_2 g \sin\theta_2$$

$$(I) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2$$
$$+ m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1$$
$$= 0$$

$$\Rightarrow m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\sin(\theta_1 - \theta_2)\, \dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\ddot{\theta}_2\right] + m_1 g l_1 \sin\theta_1$$
$$+ m_2 g l_1 \sin\theta_1 = 0$$

Divide by l_1

$$\Rightarrow m_1 l_1 \ddot{\theta}_1 + m_2 l_1 \ddot{\theta}_1 + m_2 l_2 \left[\sin(\theta_1 - \theta_2)\, \dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\ddot{\theta}_2\right] + m_1 g \sin\theta_1 + m_2 g \sin\theta_1$$
$$= 0$$

$$\Rightarrow (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \left[\sin\alpha\, \dot{\theta}_2^2 + \cos\alpha\, \ddot{\theta}_2\right] + (m_1 + m_2) g \sin\theta_1 = 0 \ \_\_\_\_\_\_\_ (4)$$

$$(I) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\sin(\theta_1 - \theta_2))(\dot{\theta}_1 - \dot{\theta}_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin\theta_2 = 0$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 l_1 l_2 (\sin(\theta_1 - \theta_2))\left(\dot{\theta}_1^2 - \dot{\theta}_1 \dot{\theta}_2\right)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin\theta_2 = 0$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 l_1 l_2 (\sin(\theta_1 - \theta_2))\dot{\theta}_1^2 + m_2 l_2 g \sin\theta_2 = 0$$

Divide by $m_2 l_2$

$$\Rightarrow l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 \dot{\theta}_1^2 (\sin(\theta_1 - \theta_2)) + g \sin\theta_2 = 0$$

$$\Rightarrow l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos\alpha - l_1 \dot{\theta}_1^2 (\sin\alpha) + g \sin\theta_2 = 0 \text{ \_\_\_\_\_\_\_\_(5)}$$

$$(4) \Rightarrow (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \left[\sin\alpha\, \dot{\theta}_2^2 + \cos\alpha\, \ddot{\theta}_2\right] + (m_1 + m_2) g \sin\theta_1 = 0$$

$$(5) \Rightarrow l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos\alpha - l_1 \dot{\theta}_1^2 (\sin\alpha) + g \sin\theta_2 = 0$$

There two equations are the required Lagrange's equation for the compound pendulum.

Non-Conservative Force Field:

The forces that do not store energy are called non-conservative force.

Eg: Friction is an example of Non-conservative force.

1. **Derive the Lagrange equation for the non-conservative force field.**

The work done by the force system in a small displacement $\delta q_1, \delta q_2, \dots, \delta q_n$ is of the form

$$\sum_{k=1}^{n} \vec{f_k} \delta \vec{r_k} = Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_n \delta q_n$$

The generalised force $q_1, q_2, \dots, q_n$ are not derivable from the potential function.

$Q_i \delta q_i$ is the work done by the force system δq_i is change to $q_i + \delta q_i$ and other q_i's are held fixed.

Consider the k^{th} particle.

Let

$$\vec{f_k} = X_k\vec{\imath} + Y_k\vec{\jmath} + Z_k\vec{k}$$

$$\vec{r_k} = x_k\vec{\imath} + y_k\vec{\jmath} + z_k\vec{k}$$

$$\delta\vec{r_k} = \delta x_k\vec{\imath} + \delta y_k\vec{\jmath} + \delta z_k\vec{k}$$

$$Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_n\delta q_n = \sum_{k=1}^{n} \vec{f_k}\delta\vec{r_k}$$

$$= \sum_{k=1}^{n} (X_k\vec{\imath} + Y_k\vec{\jmath} + Z_k\vec{k}).(\delta x_k\vec{\imath} + \delta y_k\vec{\jmath} + \delta z_k\vec{k})$$

$$= \sum_{k=1}^{n} (X_k\delta x_k + Y_k\delta y_k + Z_k\delta z_k) \_\_\_\_\_\_\_(1)$$

But X_k, Y_k, Z_k are the functions of generalised co-ordinates $q_1, q_2, \dots, q_n$

Now,

$$\delta x_k = \frac{\partial x_k}{\partial q_1}\delta q_1 + \frac{\partial x_k}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_k}{\partial q_n}\delta q_n \_\_\_\_\_\_\_(2)$$

$$\delta y_k = \frac{\partial y_k}{\partial q_1}\delta q_1 + \frac{\partial y_k}{\partial q_2}\delta q_2 + \cdots + \frac{\partial y_k}{\partial q_n}\delta q_n \_\_\_\_\_\_\_(3)$$

$$\delta z_k = \frac{\partial z_k}{\partial q_1}\delta q_1 + \frac{\partial z_k}{\partial q_2}\delta q_2 + \cdots + \frac{\partial z_k}{\partial q_n}\delta q_n \_\_\_\_\_\_\_(4)$$

Equation (1) is true for arbitrary choice of δq's, we have assume that q_i is changed to $q_i + \delta q_i$ and other q's are fixed.

Hence we have, $\delta q_1 = 0, \delta q_2 = 0, \dots, \delta q_{i-1} = 0, \delta q_{i+1} = 0, \dots, \delta q_n = 0,\ \delta q_i \neq 0$

Hence equations (2), (3) and (4) becomes

$$\delta x_k = \frac{\partial x_k}{\partial q_i}\delta q_i$$

$$\delta y_k = \frac{\partial y_k}{\partial q_i}\delta q_i$$

$$\delta z_k = \frac{\partial z_k}{\partial q_i}\delta q_i$$

$$(1) \Rightarrow Q_i\delta q_i = \sum_{k=1}^{n}\left(X_k\frac{\partial x_k}{\partial q_i}\delta q_i + Y_k\frac{\partial y_k}{\partial q_i}\delta q_i + Z_k\frac{\partial z_k}{\partial q_i}\delta q_i\right)$$

$$Q_i\delta q_i = \sum_{k=1}^{n}\left(X_k\frac{\partial x_k}{\partial q_i} + Y_k\frac{\partial y_k}{\partial q_i} + Z_k\frac{\partial z_k}{\partial q_i}\right)\delta q_i$$

$$Q_i = \sum_{k=1}^{n}\left(X_k\frac{\partial x_k}{\partial q_i} + Y_k\frac{\partial y_k}{\partial q_i} + Z_k\frac{\partial z_k}{\partial q_i}\right)$$

The result is valid whether or not the system is conservative.

The Hamilton principle for non-conservative system is

$$\delta\int_{t_1}^{t_2}(T-V)dt = 0$$

$$\Rightarrow \delta\int_{t_1}^{t_2}(T-\vec{f}\,.\vec{r})dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}(\delta T-\vec{f}\,.\delta\vec{r})dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\delta T\,dt + \int_{t_1}^{t_2}\vec{f}\,.\delta\vec{r}\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\delta T\,dt + \int_{t_1}^{t_2}(Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_n\delta q_n)\,dt = 0 \_\_\_\_\_(5)$$

Now,

$$\int_{t_1}^{t_2}\delta T\,dt = \int_{t_1}^{t_2}\left(\frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 + \cdots + \frac{\partial T}{\partial q_n}\delta q_n + \frac{\partial T}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial T}{\partial \dot{q}_2}\delta \dot{q}_2 + \cdots + \frac{\partial T}{\partial \dot{q}_n}\delta \dot{q}_n\right)dt$$

Consider

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i \, dt = \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \delta \left(\frac{\partial q_i}{\partial t}\right) dt$$

$$= \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i) dt$$

$$= \left[\frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i}\right) dt$$

$$= 0 - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i}\right) dt$$

$$(6) \Rightarrow \int_{t_1}^{t_2} \delta T \, dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_1} \delta q_1 + \frac{\partial T}{\partial q_2} \delta q_2 + \cdots + \frac{\partial T}{\partial q_n} \delta q_n\right)$$

$$- \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1}\right) \delta q_1 + \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2}\right) \delta q_2 + \cdots + \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_n}\right) \delta q_n\right] dt$$

$$= \int_{t_1}^{t_2} \left\{\left(\frac{\partial T}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1}\right)\right) \delta q_1 + \left(\frac{\partial T}{\partial q_2} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2}\right)\right) \delta q_2 + \cdots \right.$$

$$\left. + \left(\frac{\partial T}{\partial q_n} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_n}\right)\right) \delta q_n \right\} dt$$

$$(5) \Rightarrow \int_{t_1}^{t_2} \left\{\left(\frac{\partial T}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1}\right)\right) \delta q_1 + \left(\frac{\partial T}{\partial q_2} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2}\right)\right) \delta q_2 + \cdots \right.$$

$$\left. + \left(\frac{\partial T}{\partial q_n} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_n}\right)\right) \delta q_n \right\} dt + \int_{t_1}^{t_2} (Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots + Q_n \delta q_n) \, dt$$

$$= 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left\{\left(\frac{\partial T}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1}\right) + Q_1\right) \delta q_1 + \left(\frac{\partial T}{\partial q_2} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2}\right) + Q_2\right) \delta q_2 + \cdots \right.$$

$$\left. + \left(\frac{\partial T}{\partial q_n} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_n}\right) + Q_n\right) \delta q_n \right\} dt = 0$$

Hence

$$\frac{\partial T}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + Q_i = 0, \qquad i = 1,2,\dots,n$$

$$\Rightarrow \frac{\partial T}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = -Q_i, \qquad i = 1,2,\dots,n$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \qquad i = 1,2,\dots,n$$

Is valid whenever the variation of n co-ordinates $q_1, q_2, \dots, q_n$ are independent.

Unit – 3

Integral Equation

Definition 3.1:

An integral equation is an equation in which a function to be determined appears under an integral sign.

Definition 3.2:

An equation in which no non linear function of the unknown equations is involved is called linear equation.

<u>Fred – holm equation:</u>

An equation of the form

$$\propto (x) y\,(x) = F(x) + \lambda \int_a^b k\,(x,\xi) y\,(\xi) d\xi$$

Where $\propto, F$, k are given functions and λ , a , b are constants is known as Fred – holm Equation.

<u>Note:</u>

1. Here the function y(x) is to be determined
2. The function k (x, ξ) which depends upon the current variable x and auxillary variable ξ is known as Kernel of the equation.

Volterra Equation: If the upper limit of the integral is not a aconstant but is the current variable 'x', then the equation is of the form

$$\propto (x)y\,(x) = F(x) + \lambda \int_0^x k\,(x,\xi)y\,(\xi)d\xi$$

is known as Volterra Equation.

Note:

1. The constant λ could be incorrupted into kernel in both the equation
2. In many applications this constant represents a significant parameter in which we take various values.

Remark:

1. When $\propto \neq 0$, the equation involves the unknown function y both inside and outside the integral.
2. When $\propto = 0$, the unknown function appears only under the integral sign and the equation is known as an integral equation of the first kind.
3. When $\propto = 1$, the equation is said to be second kind.
4. If $\propto$ is not a constant but the prescribed function of x then, the equation is of third kind.

Note:

Equation of 3rd kind can be rewritten as equation of second kind by suitably redefining as unknown equations and the kernel

Result:

Change the integral equation of 3rd kind into the integral equation of 2nd kind.

Proof: consider the integral equation of 3rd kind. Let $\propto$ be the function of x.

Let the function $\propto (x)$ be defined in (a, b).

Let the fred – holm equation

$$\propto (x)y(x) = F(x) + \lambda \int_0^x k(x,\xi)y(\xi)d\xi$$

Divide by $\sqrt{\propto (x)}$

$$\frac{\propto (x)y(x)}{\sqrt{\propto (x)}} = \frac{F(x)}{\sqrt{\propto (x)}} + \lambda \int_a^b \frac{k(x,\xi)}{\sqrt{\propto (x)}} y(\xi)d\xi$$

$$\sqrt{\propto (x)}\mathrm{y(x)} = \frac{F(x)}{\sqrt{\propto (x)}} + \lambda \int_a^b \frac{k(x,\xi)}{\sqrt{\propto (x)}} y(\xi)d\xi$$

$$= \frac{F(x)}{\sqrt{\propto(x)}} + \lambda \int_a^b \frac{k(x,\xi)}{\sqrt{\propto(x)\propto(\xi)}} \sqrt{\propto (\xi)}y(\xi)d\xi$$

$$\mathrm{Y(x)} = \frac{F(x)}{\sqrt{\propto(x)}} + \lambda \int_a^b \frac{k(x,\xi)}{\sqrt{\propto(x)\sqrt{\propto(\xi)}}} \sqrt{\propto (\xi)}y(\xi)d\xi$$

Which is the integral equation of the 2nd kind in the unknown function

Y(x) = $\sqrt{\propto (x)}$ y(x) with modified kernel $\frac{k(x,\xi)}{\sqrt{\propto(x)\sqrt{\propto(\xi)}}}$

Note:

1. If k(x, ξ) is the symmetric function of x and ξ, then the modified kernel in the above equation given by $\frac{k(x,\xi)}{\sqrt{\propto(x)\sqrt{\propto(\xi)}}}$ preserves this symmetry.
2. Symmetric kernels are of greater importance in theory of integral equation.
3. The function $\propto (x), F(x), k(x,\xi)$ are continuous in (a,b) also, it is required that the solution y(x) is continuous in that interval.

Dimensional fred – holm equation:

Let the unknown function ω depends upon the two current variable x and y then the dimensional fred holm equation is of the form

$$\propto (x,y)\omega(x,y) = F(x,y) + \lambda \iint_R k(x,y,\xi,\eta)\omega(\xi,\eta)d\xi d\eta$$

Note:

An integral equation can be deduced from differential equation. An integral can be deduced to differential equation.

Result:

To deduce the integral equation to differential equation, we make use of the following formula

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial}{\partial x}F(x,\xi)d\xi + F(x,B(x))\frac{dB}{dx} - F(x,A(x))\frac{dA}{dx}$$

This formula is valid if both F and $\frac{\partial F}{\partial x}$ are continuous function of both x and ξ if $A'(x)$ and B' (x) are continuous.

Example:

1. consider the differentiation of the function $I_n(x)$ defined by the equation

$I_n(x) = \int_a^x (x - \xi)^{n-1} f(\xi)\ d\xi$ with F(x, ξ) = $(x - \xi)^{n-1} f(\xi)$ where n is the positive integer and 'a' is the constant.

Proof:

Given $I_n(x) = \int_a^x (x - \xi)^{n-1} f(\xi)\ d\xi$

We know that,

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial}{\partial x}F(x,\xi)d\xi + F(x,B(x))\frac{dB}{dx} - F(x,A(x))\frac{dA}{dx}$$

$$\frac{d}{dx}\left(I_n(x)\right) = \frac{d}{dx}\int_a^x (x - \xi)^{n-1} f(\xi)\ d\xi$$

$$= \int_a^x \frac{\partial}{\partial x}(x - \xi)^{n-1} f(\xi)\, d\xi + F(x,x)\frac{dx}{dx} - F(x,a)\frac{da}{dx}$$

$$= \int_a^x (n-1)(x - \xi)^{n-2} f(\xi)\, d\xi$$

$$= (n - 1)\int_a^x (x - \xi)^{n-2} f(\xi)\, d\xi$$

= (n – 1) $I_{n-1}(x)$ if n > 1 ------------- (1)

$\frac{d^2}{dx^2}\left(I_n(x)\right) = \frac{d}{dx}\left(\frac{d}{dx}(I_n(x))\right)$

$= \frac{d}{dx}((n-1)I_{n-1}(x))$

$= (n-1)\frac{d}{dx}\int_a^x (x-\xi)^{n-2} f(\xi)\, d\xi$

$= (n-1)(n-2)\int_a^x (x-\xi)^{n-3} f(\xi)\, d\xi$

$= (n-1)(n-2)\, I_{n-2}(x)$ if $n > 2$

Similarly,

$\frac{d^3}{dx^3}\left(In(x)\right) = (n-1)(n-2)(n-3)\, I_{n-3}(x)\ if\ n > 3$

….. $\frac{d^k}{dx^k}\left(I_n(x)\right) = (n-1)(n-2)\ldots(n-k)\, I_{n-k}(x)$ if $n > k$

In particular,

$\frac{d^{n-1}}{dx^{n-1}}\left(I_n(x)\right) = (n-1)(n-2)\ldots(n-(n-1))\, I_{n-(n-1)}(x)$

$= (n-1)(n-2)\ldots 2.1\, I_1(x)$

$\frac{d^{n-1}}{dx^{n-1}}\left(I_n(x)\right) = (n-1)!\ I_1(x) \ldots\ldots..(2)$

Now,

$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)\ d\xi$

$I_1(x) = \int_a^x (x-\xi)^{0} f(\xi)\ d\xi$

$I_1(x) = \int_a^x f(\xi)\ d\xi$

$\frac{d}{dx}\left(I_1(x)\right) = \frac{d}{dx}\int_a^x f(\xi)\ d\xi$

$= \int_a^x \frac{\partial}{\partial x} f(\xi)\ d\xi + F(x,x)\frac{dx}{dx} - F(x,a)\frac{da}{dx}$

$= 0 + f(x) - 0$ (since $F(x, \xi) = f(\xi)$)

$= f(x)$ ……………… (3)

From (2) we have

$$\frac{d^{n-1}(I_n(x))}{dx^{n-1}} = (n-1)!\,I_1(x)$$

$$\frac{d}{dx}\left[\frac{d^{n-1}(I_n(x))}{dx^{n-1}}\right] = (n-1)!\,\frac{d}{dx}\,(I_1(x))$$

$$\frac{d^n(I_n(x))}{dx^n} = (n-1)!\,f(x)$$

Also we have $I_n(a) = 0$ when n ≥ 1 (since upper limit and lower limit are same)

Therefore when n ≥ 1, $I_n(x)$ and its 1st (n – 1) derivative a;; vanishes when x = a.

Thus we have $I_1(x) = \int_a^x f(x_1)\,dx_1$

From (1) we have, $\frac{d}{dx}\big(I_1(x)\big)$ = (n - 1) $\big(I_{n-1}(x)\big)$

$I_n(x)$ = (n - 1) $\int_a^x I_{n-1}\,(x)dx$

$I_2(x)$ = (1) $\int_a^x I_1\,(x_2)dx_2$

$= \int_a^x \int_a^{x_2} f(x_1)dx_1dx_2$

$I_3(x)$ = 2 $\int_a^x I_2\,(x_3)dx_3$

$= 2\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1dx_2\,dx_3$

$$I_n(x) = 1.2.3\ldots.(n-1)\int_a^x \int_a^{x_n} \int_a^{x_{n-1}} \ldots.\int_a^{x_2} f(x_1)dx_1dx_2\,dx_3 \ldots dx_n$$

$$\Rightarrow \int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi = (n-1)!\int_a^x \int_a^{x_n} \int_a^{x_{n-1}} \ldots.\int_a^{x_2} f(x_1)dx_1dx_2\,dx_3 \ldots dx_n$$

$$\Rightarrow \int_a^x \int_a^{x_n} \int_a^{x_{n-1}} \ldots.\int_a^{x_2} f(x_1)dx_1dx_2\,dx_3 \ldots dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi$$

RELATION BETWEEN DIFFERENTIAL AND INTEGRAL EQUATION

1. Prove that the solution of volterra equation of 2nd kind in which the kernel k is a linear function of x can be obtained as a solution of differential equation.

Solution: consider the initial valued problem consisting of linear 2nd order differential equation.

$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$ ……… (1)

Together with prescribed initial condition

$y(a) = y_0;\ y'(a) = y_0'$…….. (2)

From (1), $y'' + A(x)y' + B(x)y = f(x)$

$\Rightarrow y'' = -A(x)y' - B(x)y + f(x)$

Integrating with respect to x_1 over the limit (a, x)

$\Rightarrow [y']_a^x = -\int_a^x A(x_1)\, y'\, dx_1 - \int_a^x B(x_1)\, y\, dx_1 + \int_a^x f(x_1)\ dx_1$

$y'(x) - y'(a) = -\int_a^x A(x_1)\, y'(x_1)\, dx_1 - \int_a^x B(x_1)\, y(x_1)\, dx_1 + \int_a^x f(x_1)\ dx_1$

$\Rightarrow\ y'(x) - y_0' = -\int_a^x A(x_1)\, y'(x_1)\, dx_1 - \int_a^x B(x_1)\, y(x_1)\, dx_1 + \int_a^x f(x_1)\ dx_1$

$\Rightarrow\ y'(x) = -\int_a^x A(x_1)\, y'(x_1) dx_1 - \int_a^x B(x_1)\, y(x_1) dx_1 + \int_a^x f(x_1)\ dx_1 + y_0'$

$= \ -[A(x_1)y'(x_1)]_a^x + \int_a^x y(x_1)\, A'(x_1) dx_1 - \int_a^x B(x_1)\, y(x_1) dx_1 + \int_a^x f(x_1)\, dx_1 + y_0'$

Integrating again we get,

$[y(x)]_a^x = -\int_a^x A(x_1)\, y(x_1) dx_1 - \int_a^x \int_a^{x_2} B(x_1) - A'(x_1) y(x_1) dx_1 dx_2 + \int_a^x \int_a^{x_2} f(x_1) dx_1 dx_2 + \int_a^x (A(a)y_0 + y_0') dx$

y(x) – y(a) = $\int_a^x A(x_1)\, y(x_1) dx_1 - \int_a^x \int_a^{x_2} B(x_1) - A'(x_1) y(x_1) dx_1 dx_2 + \int_a^x \int_a^{x_2} f(x_1) dx_1 dx_2 + (A(a)y_0 + y_0') \int_a^x dx$

y(x) $= \int_a^x A(x_1)\, y(x_1) dx_1 - \int_a^x \int_a^{x_2} B(x_1) - A'(x_1) y(x_1) dx_1 dx_2 + \int_a^x \int_a^{x_2} f(x_1) dx_1 dx_2 +$

$(A(a)y_0 + y_0')$ (x – a) + y_0

$= -\int_a^x A(\xi) y(\xi) d\xi - \frac{1}{(2-1)!}\int_a^x (x-\xi)^{2-1}[B(\xi) - A'(\xi)] y(\xi) d\xi$

$+ \frac{1}{(2-1)!}\int_a^x (x-\xi)^{2-1} f(\xi)d\xi + [A(a)y_0 + y_0'](x-a) + y_0$

$= \quad -\int_a^x [A(\xi) + (x-\xi)\big(B(\xi) - A'(\xi)\big)] y(\xi)d\xi + \int_a^x (x-\xi)^1 f(\xi)d\xi + [A(a)y_0 + y_0'](x - a) + y_0$

$= \quad \int_a^x [-A(\xi) - (x-\xi)\big(B(\xi) - A'(\xi)\big)] y(\xi)d\xi + \int_a^x (x-\xi)^1 f(\xi)d\xi + [A(a)y_0 + y_0'](x - a) + y_0$

$= \int_a^x [k(x,\xi)] y(\xi)d\xi + F(x) where\ F(x) = \int_a^x (x-\xi)^1 f(\xi)d\xi + [A(a)y_0 + y_0'](x-a) + y_0$

Which is the volterra equation of 2nd kind and the kernel k is a linear function of the current variable x.

2. Transform the $\frac{d^2y}{dx^2} + \lambda y = f(x)$ where y(0) = 1, $y'(0) = 0$ to the integral equation and conversely.

Solution:

Given $\frac{d^2y}{dx^2} + \lambda y = f(x)$ ………… (1) together with y(0) = 1, $y'(0) = 0$ …….. (2)

Equation (1) gives $y'' = -\lambda y + f(x)$

Replace x by x_1 and integrating with res. to x_1 over (0, x)

$[y']_0^x = -\lambda \int_0^x y(x_1)dx_1 + \int_0^x f(x_1)dx_1$

$y'(x) - y'(0) = -\lambda \int_0^x y(x_1)dx_1 + \int_0^x f(x_1)dx_1$

$y'(x) = -\lambda \int_0^x y(x_1)dx_1 + \int_0^x f(x_1)dx_1$

Integrating again, $[y(x)]_0^x = -\lambda \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 + \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2$

⇒ y(x) – y(0) = $-\lambda \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 + \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2$

⇒ y(x) = $-\lambda \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 + \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2 + 1$

$\Rightarrow$ y(x) = $\frac{-\lambda}{(2-1)!}\int_0^x (x-\ \xi)^{2-1} y(\xi)\ d\xi + \frac{1}{2-1}\int_0^x (x-\ \xi)^{2-1} f(\xi)\ d\xi$ + 1

$= -\lambda\ \int_0^x (x-\ \xi)^1 y(\xi)\ d\xi + \int_0^x (x-\ \xi)^{2-1} f(\xi)\ d\xi + 1$

= $\lambda\ \int_0^x (\xi - x) y(\xi)\ d\xi - \int_0^x (\xi - x)^1 f(\xi)\ d\xi + 1$ …. (3). This is the required integral equation. From this integral equation (3), we get the differential eqn (1) by the following method.

From (3), y(x) = $\lambda\ \int_0^x (\xi - x) y(\xi)\ d\xi - \int_0^x (\xi - x)^1 f(\xi)\ d\xi + 1$

$\frac{dy}{dx} = \lambda \frac{d}{dx}\int_0^x (\xi - x) y(\xi)\ d\xi - \frac{d}{dx}\int_0^x (\xi - x)^1 f(\xi)\ d\xi + \frac{d}{dx}(1)$

$= \lambda\ \left[\int_0^x \frac{\partial}{\partial x}(\xi - x) y(\xi) d\xi + (x - x) y(x)\frac{dx}{dx} - (0 - x) y(0)\frac{d(0)}{dx}\right]$

$- \int_0^x \frac{\partial}{\partial x}(\xi - x) f(\xi) d\xi + (x - x) f(x)\frac{dx}{dx} - (0 - x) f(0)\frac{d(0)}{dx} + 0$

$= \lambda\left[\int_0^x (-1) y(\xi) d\xi + 0 - 0\right] - \int_0^x (-1) f(\xi) d\xi + 0 - 0$

$= -\lambda\left[\int_0^x y(\xi) d\xi\right] + \int_0^x f(\xi) d\xi$

$\frac{d^2y}{dx^2} = -\lambda\ \frac{d}{dx}\left[\int_0^x y(\xi) d\xi\right] + \frac{d}{dx}\int_0^x f(\xi) d\xi$

$= -\lambda\ \left[\int_0^x \frac{\partial}{\partial x} y(\xi) d\xi + y(x)\frac{dx}{dx} - y(0)\frac{d(0)}{dx}\right] + \int_0^x \frac{\partial}{\partial x} f(\xi) d\xi + f(x)\frac{dx}{dx} - f(0)\frac{d(0)}{dx}$

$= -\lambda[0 + y(x) - 0] + [[0 + f(x) - 0]$

$= -\lambda\ y(x) + f(x)$

Therefore $\frac{d^2y}{dx^2} = -\lambda\ y(x) + f(x)$

3. Solve the differential eqn $y''(x) = F(x)$ with res. to the initial condition y(0) = y_0;$y'(0) = y_0'$

solution: given, $\frac{d^2y}{dx^2}$ = F(x) …. (1) together with y(0) = y_0;$y'(0) = y_0'$ … (2)

Replace x by x_1 and integrating w.r to x_1 over (0, x)

$[y']_0^x = \int_0^x F(x_1)dx_1$

$y'(x) - y'(0) = \int_0^x F(x_1)dx_1$

$y'(x) = \int_0^x y(x_1)dx_1 + y_0'$

Integrating again, $[y(x)]_0^x = \int_0^x \int_0^{x_2} F(x_1)dx_1 dx_2 + \int_0^x y_0'\, dx_1$

y(x) – y(0) = $y_0'[x-0] + \int_0^x \int_0^{x_2} F(x_1)dx_1 dx_2$

y(x) = $y_0'x + y_0 + \int_0^x \int_0^{x_2} F(x_1)dx_1 dx_2$

$= y_0'x + y_0 + \frac{1}{(2-1)!} \int_0^x (x-\xi)^{2-1} F(\xi)\, d\xi$

$= y_0'x + y_0 + \int_0^x (x-\xi)^1 F(\xi)\, d\xi$ …. (3)

This is the required integral eqn.

From the integral eqn (3), we get the differential eqn (1) by the following method.

(3) ⇒ y(x) = $y_0'x + y_0 + \int_0^x (x-\xi)^1 F(\xi)\, d\xi$

$\frac{dy}{dx} = y_0' + 0 + \frac{d}{dx}\int_0^x (x-\xi)^1 F(\xi)\, d\xi$

$= y_0' + \int_0^x \frac{\partial}{\partial x}(x-\xi)^1 F(\xi)\, d\xi + (x-x)F(x)\frac{dx}{dx} - (x-0)F(0)\frac{d(0)}{dx}$

$= y_0' + \int_0^x (F(\xi)\, d\xi$

$\frac{d^2y}{dx^2} = 0 + \frac{d}{dx}\int_0^x (F(\xi)\, d\xi$

$= \int_0^x \frac{\partial}{\partial x} F(\xi)d\xi + F(x)\frac{dx}{dx} - F(0)\frac{d(0)}{dx}$

= 0 + F(x) – 0

$\frac{d^2y}{dx^2} = F(x)$

4. Solve the differential eqn $y''(x) = F(x)$ where y satisfies the end condition y(0) = 0; y(1) = 0

soln: $y''(x) = F(x)$ …. (1) together with the end condition y(0) = 0; y(1) = 0 …. (2)

Replace x by x_1 and integrating w.r to x_1 over (0, x)

$[y']_0^x = \int_0^x F(x_1)dx_1$

$y'(x) - y'(0) = \int_0^x F(x_1)dx_1$

$y'(x) = \int_0^x y(x_1)dx_1 + y'(0)$

Integrating again, $[y(x)]_0^x = \int_0^x \int_0^{x_2} F(x_1)dx_1 + \int_0^x y'(0)dx_1$

y(x) – y(0) = $\frac{1}{(2-1)!} \int_0^x (x - \xi)^1 F(\xi)\ d\xi + y'(0)x$

y(x) = $\int_0^x (x - \xi)^1 F(\xi)\ d\xi + y'(0)x$ …. (3)

put x = 1

y(1) = $\int_0^1 (1 - \xi)^1 F(\xi)\ d\xi + y'(0)$

$\Rightarrow \int_0^1 (1 - \xi)^1 F(\xi)\ d\xi = -y'(0)$

$\Rightarrow y'(0) = - \int_0^1 (1 - \xi)^1 F(\xi)\ d\xi$

$= \int_0^1 (\xi - 1)^1 F(\xi)\ d\xi$

(3) ⇒ y(x) = $\int_0^x (x - \xi)^1 F(\xi)\ d\xi + x \int_0^1 (\xi - 1)^1 F(\xi)\ d\xi$

$= \int_0^x (x - \xi)^1 F(\xi)\ d\xi + x \int_0^x (\xi - 1)^1 F(\xi)\ d\xi + x \int_x^1 (\xi - 1)^1 F(\xi)\ d\xi$

$= \int_0^x [(x - \xi)^1 + x(\xi - 1)^1] F(\xi)\ d\xi + x \int_x^1 (\xi - 1)^1 F(\xi)\ d\xi$

$= \int_0^x [x - \xi + x\xi - x] F(\xi)\ d\xi + \int_x^1 (x\xi - x)\ F(\xi)\ d\xi$

y(x) = $\int_0^x (x\xi - \xi) F(\xi) d\xi + \int_x^1 (x\xi - x)\ F(\xi)\ d\xi$ …. (3)

This is the required integral eqn.

The above eqn is of the form,

y(x) = $\int_0^x k(x,\xi)F(\xi)d\xi$ where k(x, ξ) = $\begin{Bmatrix} x\xi - \xi, \xi < x \\ x\xi - x, \xi > x \end{Bmatrix}$

(3) ⇒ y(x) = $\int_0^x (x\xi - \xi)F(\xi)d\xi + \int_x^1 (x\,\xi - x) \;\; F(\xi)\; d\xi$

$y'(x) = \frac{d}{dx}\int_0^x (x\xi - \xi)F(\xi)\; d\xi + \frac{d}{dx}\int_0^1 (x\xi - x)^1 F(\xi)\; d\xi$

= $\int_0^x \frac{\partial}{\partial x}(x\xi - \xi)F(\xi)\; d\xi + (x^2 - x)F(x)\frac{dx}{dx} - (x.0 - 0)F(0)\frac{d(0)}{dx} + \int_0^1 \frac{\partial}{\partial x}(x\xi - x)^1 F(\xi)\; d\xi$

$$+(x^2 - x)F(x)\frac{dx}{dx} - (x^2 - x)F(x)\frac{d(x)}{dx}$$

= $\int_0^x \xi F(\xi)\; d\xi + (x^2 - x)F(x) + \int_x^1 (\xi - 1) \;\; F(\xi)\; d\xi + 0 - (x^2 - x)F(x)$

= $\int_0^x \xi F(\xi)\; d\xi + \int_x^1 (\xi - 1) \;\; F(\xi)\; d\xi$

$y''(x) = \frac{d}{dx}\int_0^x \xi F(\xi)\; d\xi + \frac{d}{dx}\int_x^1 (\xi - 1) \;\; F(\xi)\; d\xi$

$= \int_0^x \frac{\partial}{\partial x}(\xi)F(\xi)\; d\xi + (x)F(x)\frac{dx}{dx} - (0)F(0)\frac{d(0)}{dx} + \int_0^1 \frac{\partial}{\partial x}(\xi - 1)^1 F(\xi)\; d\xi$

$$+(1 - 1)F(1)\frac{d(1)}{dx} - (x - 1)F(x)\frac{d(x)}{dx}$$

= 0 + xF(x) + 0 + 0 – (x – 1)F(x)

= F(x)

5. Solve $\frac{d^2y}{dx^2} + \lambda y = 0, y(0) = 0, y(l) = 0$

Soln:

$\frac{d^2y}{dx^2} + \lambda y = 0$ ….. (1)

$\Rightarrow y'' + \lambda y = 0$

Replace x by x_1 and integrating with res. to x_1 over (0, x)

$\Rightarrow [y']_0^x + \lambda \int_0^x y\, dx_1 = 0$

$[y'(x) - y'(0)] + \lambda \int_0^x y\, dx_1 = 0$

Again integrating, $[y(x)]_0^x = \int_0^x y'(0)\, dx_1 - \lambda \int_0^x \int_0^{x_2} y\, dx_1 dx_2$

$y(x) - y(0) = -\frac{\lambda}{(2-1)!} \int_0^x (x - \xi)^{2-1} y(\xi)\, d\xi + y'(0)(x - 0)$

$y(x) = xy'(0) - \lambda \int_0^x (x - \xi)^1 y(\xi)\, d\xi$

put $x = l$,

$y(l) = ly'(0) - \lambda \int_0^l (l - \xi)^1 y(\xi)\, d\xi$

$0 = ly'(0) - \lambda \int_0^l (l - \xi)^1 y(\xi)\, d\xi$

$y'(0) = \frac{\lambda}{l} \int_0^l (l - \xi)^1 y(\xi)\, d\xi$

$y(x) = x\frac{\lambda}{l} \int_0^l (l - \xi)^1 y(\xi)\, d\xi - \lambda \int_0^x (x - \xi)^1 y(\xi)\, d\xi$

$= x\frac{\lambda}{l}\left[\int_0^x (l - \xi)^1 y(\xi)\, d\xi + \int_x^l (l - \xi)^1 y(\xi)\, d\xi\right] - \lambda \int_0^x (x - \xi)^1 y(\xi)\, d\xi$

$= \lambda \int_0^x [\frac{(l-\xi)^1 x}{l} - (x - \xi)^1]\, y(\xi)\, d\xi + x\frac{\lambda}{l} \int_x^l (l - \xi)^1 y(\xi)\, d\xi$

The required integral eqn is

$= \lambda \int_0^x \frac{lx - \xi x \ \ - lx + l\xi}{l}\, y(\xi)\, d\xi + x\frac{\lambda}{l} \int_x^l (l - \xi)^1 y(\xi)\, d\xi$

$= \lambda \int_0^x \frac{(l-x)\xi}{l}\, y(\xi)\, d\xi + x\frac{\lambda}{l} \int_x^l (l - \xi)^1 y(\xi)\, d\xi$

$\Rightarrow y(x) = \lambda \int_0^l k(x,\xi)\, y(\xi)\, d\xi \text{ where } k(x,\xi) = \begin{Bmatrix} \frac{(l-x)\xi}{l}, \xi < x \\ \frac{(l-\xi)x}{l}, \xi > x \end{Bmatrix}$

Diff w.res.to x, we get

$$y'(x) = \lambda \frac{d}{dx}\int_0^x \frac{\xi}{l}(l-x)y(\xi)\ d\xi + \frac{d}{dx}\ \lambda \int_x^l \frac{x}{l}(l-\xi)\ \ y(\xi)\ d\xi$$

$$= \lambda \int_0^x \frac{\partial}{\partial x}[\frac{\xi}{l}(l-x)y(\xi)]d\xi + \frac{\lambda x}{l}(l-x)y(x)\frac{dx}{dx} - (0) + \lambda \int_x^l \frac{\partial}{\partial x}(\frac{x}{l})\ (l-\xi)^1 y(\xi)\ d\xi$$

$$+\frac{\lambda x}{l}(l-\ l)y(l)\frac{d(l)}{dx} - \frac{\lambda x}{l}(l-x)y(x)\frac{d(x)}{dx}$$

$$= \quad \lambda \int_0^x -\frac{\xi}{l}y(\xi)d\xi + \frac{\lambda x}{l}(l-x)y(x) + \lambda \int_x^l (\frac{1}{l})\ (l-\xi)^1 y(\xi)\ d\xi + \frac{\lambda x}{l}(l-\ l)y(l)\frac{d(l)}{dx} - \frac{\lambda x}{l}(l-x)y(x)$$

$$= -\lambda \int_0^x \frac{\xi}{l}y(\xi)d\xi + \lambda \int_x^l (\frac{1}{l})\ (l-\xi)^1 y(\xi)\ d\xi$$

$$y''(x) = -\lambda \frac{d}{dx}\ \int_0^x \frac{\xi}{l}y(\xi)\ d\xi +\ + \frac{\lambda}{l}\frac{d}{dx}\int_x^l (l-\ \xi)\ \ y(\xi)\ d\xi$$

$$= -\lambda\left\{\int_0^x \frac{\partial}{\partial x}\frac{\xi}{l}y(\xi)\ d\xi + \frac{x}{l}y(x) - 0\right\} + \qquad \frac{\lambda}{l}\int_x^l \frac{\partial}{\partial x}(l-\xi)^1 y(\xi)\ d\xi + (l-\ l)y(l) - (l-x)y(x)$$

$$= -\lambda\left[\frac{x}{l}y(x)\right] + \frac{\lambda}{l}\ [(x-l)y(x)]$$

$= 0 - \lambda y(x)$

$y''(x) + \ \lambda y(x) = 0$

To find the boundary condition, we use the integral eqn

$$y(x) = \lambda\ \int_0^x \frac{(l-x)\xi}{l}\ y(\xi)\ d\xi +\ x\frac{\lambda}{l}\int_x^l (l-\ \xi)^1 y(\xi)\ d\xi$$

put x = 0, y(0) = 0

put x = l, y(l) = 0

The kernel of the above D.E has two expression in the region $\xi < x$ and $\xi > x$.

But the expressionis equivalent when $\xi = x$

We have, $k(x, \xi) = \begin{cases} \frac{(l-x)\xi}{l}, \xi < x \\ \frac{(l-\xi)x}{l}, \xi > x \end{cases}$

Put $\xi = x$, k(x, x) = $\begin{Bmatrix} \frac{(l-x)x}{l} \\ \frac{(l-x)x}{l} \end{Bmatrix}$

If we think of k as a function of x for a fixed value of ξ, k is continuous at $\xi = x$

k(x, ξ) = $\begin{Bmatrix} \frac{(l-x)\xi}{l}, \xi < x \\ \frac{(l-\xi)x}{l}, \xi > x \end{Bmatrix}$

if k is a linear function of x and then k satisfies the D.E $\frac{\partial^2 k}{\partial x^2} = 0$ and k vanishes at the end point is x = 0 and x = l

finally we notice that k(x, ξ) is unchanged if x and ξ are interchanged.

Ie) k(x, ξ) = k(ξ, x)

Kernels having this symmetry property are called symmetric kernels.

6. Solve y'' + Ay' + By = 0, y(0) = y(1) = 0. Also find kernel.

Solution: y'' + Ay' + By = 0, y(0) = y(1) = 0

Replace x by x_1 and integrating w.res.to x_1 over (0, x)

$[y']_0^x = -\int_0^x Ay'\, dx_1 - \int_0^x By\, d\, x_1$

$\Rightarrow y'(x) - y'(0) = -A[y]_0^x - \int_0^x By\, dx_1$

$y'(x) = y'(0) - A[y]_0^x - \int_0^x By\, dx_1$

again integrating,

$[y(x)]_0^x = \int_0^x y'(0)\, dx_1 - A\int_0^x y\, dx_1 - \int_0^x \int_0^{x_2} By\, dx_1 dx_2$

y(x) – y(0) = $y'(0)(x-0) - A\int_0^x y\, dx_1 - \frac{B}{(2-1)!}\int_0^x (x-\xi)^{2-1} y(\xi)\, d\xi$

y(x) = $y'(0)(x) - \frac{A}{(1-1)!}\int_0^x (x-\xi)^{1-1} y(\xi)\, d\xi - B\int_0^x (x-\xi)^1 y(\xi)\, d\xi$

y(x) = $y'(0)(x) - A\int_0^x y(\xi)\ d\xi - B\int_0^x (x-\ \xi)^1 y(\xi)\ d\xi$

Put x = 1,

y(1) = $y'(0)(1) - \int_0^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi$

$y'(0) = \int_0^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi$

y(x) = $(x)\int_0^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi\ - A\int_0^x y(\xi)\ d\xi - B\int_0^x (x-\ \xi)^1 y(\xi)\ d\xi$

= $(x)\int_0^x A + B\,(1-\ \xi)^1 y(\xi) d\xi +\ (x)\int_x^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi - (x)\int_0^x A + B\,(x - \xi)^1 y(\xi) d\xi$

= $\int_0^x [xA + B\,x(1-\ \xi)^1 - A - B(x-\ \xi)^1] y(\xi) d\xi +\ (x)\int_x^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi$

y(x) =$\int_0^x [A(x-1) + (\ \xi)^1(-Bx +\ B)] y(\xi) d\xi +\ (x)\int_x^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi$

therefore, y(x) = $\int_x^1 k(x,\xi)\, y(\xi) d\xi$

where $k(x,\xi) = \begin{Bmatrix} [A(x-1) + (\ \xi)^1(-Bx +\ B)]\,, \xi < x \\ Ax + Bx(1-\xi), \xi > x \end{Bmatrix}$

diff w.res.to x, we get

y'(x) = $\frac{d}{dx}\int_0^x [A(x-1) + (\ \xi)^1(-Bx +\ B)] y(\xi) d\xi + \frac{d}{dx}(x)\int_x^1 A + B\,(1-\ \xi)^1 y(\xi) d\xi$

= $\int_0^x \frac{\partial}{\partial x}[A(x-1) + (\ \xi)^1(-Bx +\ B)] y(\xi) d\xi + [A\,(x-1) +\ xB(1-x)] y(x)\frac{dx}{dx}$

$-A(x-1) +\ 0.B(1-x)y(0)\ \frac{d(0)}{dx}\quad + \quad \int_x^1 \frac{\partial}{\partial x}(Ax + Bx - Bx\xi) y(\xi) d\xi + [A\,x +\ xB(1 - 1)] y(1)\frac{d(1)}{dx}$

$$-[A(x) +\ Bx(1-x)y(x)\ \frac{d(x)}{dx}$$

= $\int_0^x (A - B\xi) y(\xi) d\xi + [Ax - A + Bx - Bx^2] y(x) + \int_x^1 (A + B - B\xi) y(\xi) d\xi - [Ax + Bx - Bx^2] y(\xi)$

= $\int_0^x (A - B\xi) y(\xi) d\xi -\ Ay(x) +\ \int_x^1 (A + B - B\xi) y(\xi) d\xi$

Again diff with res. to x, we get

$$y''(x) = \frac{d}{dx}\int_0^x (A - B\xi)y(\xi)d\xi - Ay'(x) + \frac{d}{dx}\int_x^1 (A + B - B\xi)y(\xi)d\xi$$

$$= \int_0^x \frac{\partial}{\partial x}(A - B\xi)y(\xi)d\xi + (A - Bx)y(x)\frac{dx}{dx} - \left(A - B(0)y(0)\frac{d(0)}{dx}\right)$$

$$+ \int_X^1 \frac{\partial}{\partial x}(A + B - B\xi)y(\xi)d\xi + (A + B - B)y(1)\frac{d(1)}{dx} - (A + B - Bx)y(x)\frac{dx}{dx} - Ay'(x)$$

$$= 0 + (A - Bx)y(x) - (A + B - Bx)y(x) - Ay'(x)$$

$$y''(x) = -Ay'(x) - By(x)$$

$$y''(x) + Ay'(x) + By(x) = 0$$

$$k(x,\xi) = \begin{cases} [A(x-1) + (B\xi)^1(1-x)], \xi < x \\ Ax + Bx(1-\xi), \xi > x \end{cases}$$

The kernel obtained in this way is a non symmetric kernel and it is discontinuous at x = $\xi$where A = 0

If A= 0, the kernel is,

$$k(x,\xi) = \begin{cases} B\xi)^1(1-x), \xi < x \\ Bx(1-\xi), \xi > x \end{cases}$$

Here the kernel k is symmetric. The kernel k is non symmetric but if A = 0,the kernel becomes symmetric.

<u>Green's function:</u>

Consider the differential equation $Ly + \emptyset(x) = 0$

Where L is the differential operator,

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

$$= p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

Together with homogenous boundary condition

$\alpha y + \beta \frac{dy}{dx} = 0$

Where α *and* β are constants and the conditions are imposed at the end points a$\leq x \leq b$

Ie) the homogenous boundary conditions are

$$\alpha\, y(a) + \beta \left(\frac{dy}{dx}\right)_{x=a} = 0$$

$$\alpha\, y(b) + \beta \left(\frac{dy}{dx}\right)_{x=b} = 0$$

$\emptyset$ may be a function of x or depends upon x by the unknown function y(x) given by

$\emptyset = \emptyset(x, y(x))$

Let us reformulate the problem. Let us determine green function G which for a given number ξ is given by

$G_1(x)\ where\ x < \xi\ , G_2(x) where\ x > \xi$

And which has the following 4 properties

1. The function $G_1 and\ G_2$ satisfy the equation L(G) = 0 in the intervals of definition.
   Ie) $L(G_1) = 0\ , x < \xi,\ L(G_2) = 0, x > \xi$
2. The function G satisfies the homogeneous condition prescribed at the end points x = a and x = b
   Ie) G_1 satisfies the prescribed condition at x = a; G_2 satisfies the prescribed condition at x = b
3. The function G is continuous at x = ξ, ie) $G_1(\xi) = G_2(\xi)$
4. The derivative of G has a discontinuous magnitude $-\frac{1}{\emptyset(\xi)}$ at the point x = ξ
   Ie) $G_2'(\xi) - G_1'(x) = -\frac{1}{p(\xi)}$

Note: here it may be assumed that the function p(x) is continuous and p(x) ≠ 0 inside the interval (a, b) so that the discontinuity of the derivative of G is of finite magnitude. Similarly p(x) and q(x) are continuous at (a, b)

ABEL'S FORMULA:

Let u(x) and v(x) satisfies L(y) = 0, then [A = P(uv' – u'v)]' = 0

Proof:

$$L = \frac{pd^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

$$Ly = p\frac{d^2}{dx^2} + \frac{\frac{dp}{dx}dy}{dx} + qy$$

$$Ly = \left(p\frac{dy}{dx}\right)' + dy \;\ldots\ldots (1)$$

Given u(x) and v(x) satisfy Ly = 0

$$Lu = (pu')' + qu; \;\; Lv = (pv')' + qv$$

$$ie.)\ (pu')' + qu = 0 \ldots .(2)\ (since\ Lu = 0\ and\ Lv = 0)$$

$$(pv')' + qv = 0 \ldots .(3)$$

Multiply (2) by v and (3) by u,

$$\Rightarrow (pu')'v - (pv')'u = 0$$

$$\Rightarrow \ (pv')'u - (pu')'v = 0$$

$$\Rightarrow (pv'' + p'v')u - (pu'' + p'u')v = 0$$

$$\Rightarrow puv'' + p'uv' - pu''v - p'u'v = 0$$

$$\Rightarrow p(uv'' - u''v + u'v' - u'v') + p'(uv' - u'v) = 0$$

$$\Rightarrow p\frac{d}{dx}(uv' - u'v) + p'(uv' - u'v) = 0$$

$$\frac{d}{dx}\left(p(uv' - u'v)\right) = 0$$

$$\Rightarrow [p(uv' - u'v)]' = 0\ gives\ A' = 0$$

Here A is called Abel's formula given by A = $p(uv' - u'v)$

1. Show that the function G(x, ξ) exists the original problem can be transformed to the relation $y(x) = \int_a^b G(x,\xi)\emptyset(\xi)d\xi \ldots.$ (A). The equation (A) defines the solution of the problem when ∅ is the direct function of x. also eqn (A) an equivalent integral eqn of the problem when ∅ involves y.

Solution: Let y = u(x) be a non trivial solution of the eqn L(y) = 0 which satisfies the prescribed homogeneous boundary condition at x = a ….. (1)

Let y = v(x) be a non – trivial solution of the eqn L(y) = 0 which satisfies the prescribed homogeneous boundary condition at x = b … (2)

The condition (1) and (2) are satisfied if

$$G_1(x,\xi) = \begin{Bmatrix} c_1u(x), x < \xi \\ c_2v(x), x > \xi \end{Bmatrix} \ldots..(3)$$

To prove: G_1 *and* G_2 satisfies the four condition of the Green's function.

(i) To prove : $L(G_1) = 0$ *and* $L(G_2) = 0$

Since L(y) = 0; L(u(x)) = 0, L(v(x)) = 0

Let x < ξ

Then , $L(G_1) = L(c_1u(x)) = c_1L(u(x)) = 0$

Let x > ξ

Then , $L(G_2) = L(c_2v(x)) = c_2L(v(x)) = 0$

Therefore G satisfies the 1^{st} condition.

(ii) To prove: G_1 *and* G_2 satisfies the boundary condition.

By our choice of u(x), u(x) satisfies the homogeneous end condition at x = a.

Ie) $\alpha\, u(x) + \beta \frac{du(x)}{dx} = 0$ at x = a

Multiply by c_1,

$$c_1\alpha\, u(x) + c_1\beta \frac{d}{dx}\big(u(x)\big) = 0 \; at \; x = a$$

$$\Rightarrow \alpha(c_1u(x)) + \beta \frac{d}{dx}\big(c_1u(x)\big) = 0 \; at \; x = a$$

$\Rightarrow \alpha(G_1) + \beta \frac{d}{dx}(G_1) = 0 \; at \; x = a$

similarly, $\alpha(G_2) + \beta \frac{d}{dx}(G_2) = 0 \; at \; x = b$

Hence G satisfies the 2nd condition.

(iii) To prove: $G_1 \; and \; G_2$ are continuous at x = ξ. For the function $G_1 \; and \; G_2$ satisfies the 3rd condition, we must have $G_1(\xi) = G_2(\xi)$

$G_1(\xi) - G_2(\xi) = 0$

$c_1 u(\xi) - c_2 v(\xi) = 0$

$-c_1 u(\xi) + c_2 v(\xi) = 0 \ldots. (4)$

For the function $G_1 \; and \; G_2$ to satisfies the 4th condition we have,

$G_1'(\xi) - G_2'(\xi) = -\frac{1}{p(\xi)}$

$-c_1 u'(\xi) + c_2 v'(\xi) = -\frac{1}{p(\xi)} \ldots. (5)$

Solving (4) and (5), we get,

$$\begin{vmatrix} v(\xi) & u(\xi) \\ v'(\xi) & u'(\xi) \end{vmatrix} \neq 0$$

$\Rightarrow v(\xi)u'(\xi) - u(\xi)v'(\xi) \neq 0$

$\Rightarrow u(\xi)v'(\xi) - v(\xi)u'(\xi) \neq 0$

This quantity is determined by

$$W(u(\xi), v(\xi)) = \begin{vmatrix} v(\xi) & u(\xi) \\ v'(\xi) & u'(\xi) \end{vmatrix} \neq 0$$

This is called as Wronskian of the solution u and v of the equation L(y) = 0.

If suppose u and v are linearly dependent, then $u(\xi) = k \, v(\xi)$ where k is a constant.

$\Rightarrow \frac{u(\xi)}{v(\xi)} = k$

$\Rightarrow \frac{v(\xi) \, u'(\xi) - u(\xi) \, v'(\xi)}{[v(\xi)]^2} = 0$

$\Rightarrow v(\xi) \, u'(\xi) - u(\xi) \, v'(\xi) = 0$

$\Rightarrow u(\xi) \, v'(\xi) - v(\xi) \, u'(\xi) = 0$

$\Rightarrow W(u(\xi), v(\xi)) = 0$

Therefore the Wronskian $W(u(\xi), v(\xi))$ cannot vanish unless the function are linearly dependent.

By Abel's formula we have,

$W(u(\xi), v(\xi)) = u(\xi)\, v'(\xi) - v(\xi)\, u'(\xi) = \frac{A}{p(\xi)}$......(6) where A is constant independent.

From (5) and (6) we have

(5) $\Rightarrow -c_1 u'(\xi) + c_2 v'(\xi) = -\frac{1}{p(\xi)}$

(6) $\Rightarrow u(\xi)\, v'(\xi) - v(\xi)\, u'(\xi) = \frac{A}{p(\xi)}$

Multiply (5) by – A

$\Rightarrow c_1 A u'(\xi) - c_2 A v'(\xi) = \frac{A}{p(\xi)}$ ....(7)

Comparing eqn (6) and (7), we get

$u(\xi) = -c_2 A; -v(\xi) = c_1 A$

$\Rightarrow c_2 = -\frac{u(\xi)}{A}; c_1 = -\frac{v(\xi)}{A}$

Therefore (3) $\Rightarrow G(x, \xi) = \begin{Bmatrix} -\frac{1}{A} u(x) v(\xi), x < \xi \\ -\frac{1}{A} v(x) u(\xi), x > \xi \end{Bmatrix}$

This G satisfies both the condition (3) and (4) in Green's function. This G is the required Green's function.

G does not exist $\Leftrightarrow$ A vanish

$$\Leftrightarrow W(u(\xi), v(\xi)) \text{ vanishes}$$

$$\Leftrightarrow W(u(\xi), v(\xi)) = 0$$

$$\Leftrightarrow u(\xi)\, v'(\xi) - v(\xi)\, u'(\xi) = 0$$

$$\Leftrightarrow \frac{v'(\xi)}{v(\xi)} = \frac{u'(\xi)}{u(\xi)}$$

$$\Leftrightarrow \log(v(\xi)) = \log(u(\xi)) + \log k$$

$$\Leftrightarrow \log(v(\xi)) - \log(u(\xi)) = \log k$$

$$\Leftrightarrow \log\left(\frac{v(\xi)}{u(\xi)}\right) = \log k$$

$$\Leftrightarrow \frac{v(\xi)}{u(\xi)} = k$$

$$\Leftrightarrow ku(\xi) = v(\xi)$$

$\Leftrightarrow$ u and v are linearly dependent.

2. Show that the relation y(x) = $\int_a^b G(x,\xi)\emptyset(\xi)d\xi$

Where $G(x,\xi) = \begin{Bmatrix} -\frac{1}{A}u(x)v(\xi)\,, x < \xi \\ -\frac{1}{A}v(x)u(\xi)\,, x > \xi \end{Bmatrix}$ implies the differential eqn Ly + $\emptyset(x) = 0$ together with the prescribed boundary condition.

Soln: given y(x) = $\int_a^b G(x,\xi)\emptyset(\xi)d\xi$

Where $G(x,\xi) = \begin{Bmatrix} -\frac{1}{A}v(x)u(\xi)\,, x > \xi \\ -\frac{1}{A}u(x)v(\xi)\,, x < \xi \end{Bmatrix}$

y(x) = $\int_a^x -\frac{1}{A}v(x)u(\xi)\,\emptyset(\xi)d\xi - \int_x^b \frac{1}{A}u(x)v(\xi)\,\emptyset(\xi)d\xi$

diff. w.res.to x,

$$\frac{dy}{dx} = -\frac{1}{A}\frac{d}{dx}\int_a^x v(x)u(\xi)\,\emptyset(\xi)d\xi - \frac{1}{A}[u(x)v(x)\emptyset(x)]\frac{dx}{dx} + \frac{1}{A}[v(x)u(a)\emptyset(a)]\frac{da}{dx}$$

$$-\frac{1}{A}\frac{d}{dx}\int_x^b u(x)v(\xi)\,\emptyset(\xi)d\xi + \frac{1}{A}[u(x)v(x)\emptyset(x)]\frac{dx}{dx} - \frac{1}{A}[u(x)v(b)\emptyset(b)]\frac{db}{dx}$$

$$= -\frac{1}{A}\int_a^x \frac{\partial}{\partial x}v(x)u(\xi)\,\emptyset(\xi)d\xi - \frac{1}{A}[u(x)v(x)\emptyset(x)] + 0$$

$$-\frac{1}{A}\int_x^b \frac{\partial}{\partial x}u(x)v(\xi)\,\emptyset(\xi)d\xi - 0 + \frac{1}{A}[u(x)v(x)\emptyset(x)]$$

$$= -\frac{1}{A}\int_a^x v'(x)u(\xi)\,\emptyset(\xi)d\xi - \frac{1}{A}\int_x^b u'(x)v(\xi)\,\emptyset(\xi)d\xi$$

Again diff with respect to x,

$$\frac{d^2y}{dx^2} = -\frac{1}{A}\int_a^x \frac{\partial}{\partial x}(v'(x)u(\xi)\, \emptyset(\xi))d\xi \; -\frac{1}{A}[u(x)v'(x)\emptyset(x)]\frac{dx}{dx} + \frac{1}{A}[v'(x)u(a)\emptyset(a)]\frac{da}{dx}$$

$$-\frac{1}{A}\int_x^b \frac{\partial}{\partial x}(u'(x)v(\xi)\, \emptyset(\xi))d\xi + \frac{1}{A}[u'(x)v(x)\emptyset(x)]\frac{dx}{dx} - \frac{1}{A}[u'(x)v(b)\emptyset(b)]\frac{db}{dx}$$

$$= -\frac{1}{A}\int_a^x v''(x)u(\xi)\, \emptyset(\xi))d\xi - \frac{1}{A}[u(x)v'(x)\emptyset(x)] + 0$$

$$-\frac{1}{A}\int_x^b (u''(x)v(\xi)\, \emptyset(\xi))d\xi + \frac{1}{A}[u'(x)v(x)\emptyset(x)]$$

Now, Ly(x) = p(x)y''(x) + p'(x)y'(x) + q(x)y(x) (since L = $\frac{pd^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$)

$$p(x)y''(x) + p'(x)y'(x) + q(x)y(x)$$

$$= -\frac{p(x)}{A}\left[\int_a^x v''(x)u(\xi)\, \emptyset(\xi))d\xi + \int_x^b (u''(x)v(\xi)\, \emptyset(\xi))d\xi\right] + u(x)v'(x)\emptyset(x)$$

$$- u'(x)v(x)\emptyset(x)$$

$$-\frac{p(x)}{A}\left[\int_a^x v''(x)u(\xi)\, \emptyset(\xi))d\xi + \int_x^b (u''(x)v(\xi)\, \emptyset(\xi))d\xi\right]$$

$$-\frac{q(x)}{A}\left[\int_a^x v(x)u(\xi)\, \emptyset(\xi))d\xi + \int_x^b (u(x)v(\xi)\, \emptyset(\xi))d\xi\right]$$

Ly(x) = $-\frac{1}{A}\left[\int_a^x (p(x)v''(x) + p'(x)v'(x) + q(x)v(x))u(\xi)\, \emptyset(\xi)d\xi\right]$

$-\frac{1}{A}\left[\int_x^b (u''(x)p(x) + p'(x)u'(x) + q(x)u(x))\, v(\xi)\, \emptyset(\xi))d\xi\right]$

$-\frac{1}{A}[p(x)[u(x)v'(x) - v(x)u'(x)]\emptyset(x)]$

$= -\frac{1}{A}\left[\int_a^x (L(v(x))u(\xi)\, \emptyset(\xi)d\xi\right]$

$-\frac{1}{A}\left[\int_x^b (L(u(x))\, v(\xi)\, \emptyset(\xi))d\xi\right]$

$-\frac{1}{A}[A(\emptyset(x)] = 0 + 0 - \emptyset(x)$ (since Abel's formula)

Since Ly(x) = $-\emptyset(x) \Rightarrow$ Ly(x) $- \emptyset(x) = 0$

Hence the function y(x) satisfies the differential equation.

Now y(a) = $-\frac{1}{A}[\int_a^b u(a)\, v(\xi)\, \emptyset(\xi)d\xi$

y(a) = $-\frac{u(a)}{A}\int_a^b\, v(\xi)\, \emptyset(\xi)d\xi$

y'(a) = $-\frac{u'(a)}{A}\int_a^b\, v(\xi)\, \emptyset(\xi)d\xi$

To prove: y(x) satisfies the condition at x = a and x = b.

Ie) to prove: $[\alpha y+\beta y']_{x=a}=0$

We know that, u(x) and v(x) satisfies the condition at x = a and x = b.

$[\alpha u+\beta u']_{x=a}=0$; $[\alpha v+\beta v']_{x=b}=0$

Now, $[\alpha u(a)+\beta u'(a)] \quad = 0$

Multiply by $-\frac{1}{A}[\int_a^b\, v(\xi)\, \emptyset(\xi)d\xi \Rightarrow -\frac{\alpha\, u(a)}{A}\int_a^b\, v(\xi)\, \emptyset(\xi)d\xi - \; -\frac{\beta\, u'(a)}{A}\int_a^b\, v(\xi)\, \emptyset(\xi)d\xi = 0$

$\Rightarrow \alpha y(a)+\beta y'(a)=0$

$[\alpha y+\beta y']_{x=a}=0$. Similarly, $[\alpha y+\beta y']_{x=b}=0$

Therefore y(x) satisfies the boundary condition at x = a and x = b.

<u>Particular case:</u>

Let $\emptyset(x) = \lambda r(x)y(x) - f(x)$ then the equivalent differential eqn Ly + $\emptyset(x)=0$

$\Rightarrow L(x)+\ \lambda r(x)y(x)-f(x) = 0$

$L(x)+\ \lambda r(x)y(x)=f(x)$ with the associated homogenous condition imposed at the endpoint of (a, b).

Hence the corresponding Fred – Holm Eqn is of the form

y(x) = $\lambda\int_a^b G(x,\xi)\, r(\xi)\, y(\xi)d\xi$ − y(a) = $-\int_a^b G(x,\xi)\, f(\xi)d\xi$ ....(1) where G is the relevant Green's function. Here the kernel k(x, ξ) is a product of G(r, ξ)r(ξ) and G(x, ξ) is symmetric. If r(x) is not a constant, then the product k(x, ξ) is not symmetric. In this case, by assuming r(x) as a non-negative integer over (a, b), we can write, $y(x) = \sqrt{r(x)}\, y(x)$

Multiply (1) by $r(x) \Rightarrow$ y(x) $\sqrt{r(x)}$ $= \lambda \int_a^b G(x,\xi)\, \sqrt{r(x)} r(\xi)\, y(\xi) d\xi - \int_a^b G(x,\xi) \sqrt{r(x)}\, f(\xi) d\xi$

$= \lambda \int_a^b G(x,\xi)\, \sqrt{r(x)} \sqrt{r(\xi)} \sqrt{r(\xi)}\, y(\xi) d\xi - \int_a^b \frac{G(x,\xi)\sqrt{r(x)}\,\sqrt{r(\xi)}}{\sqrt{r(\xi)}}\, f(\xi) d\xi$

$= \lambda \int_a^b G(x,\xi) \sqrt{r(x)} \sqrt{r(\xi)} V(\xi)\, d\xi - \int_a^b \frac{G(x,\xi)\sqrt{r(x)}\,\sqrt{r(\xi)}}{\sqrt{r(\xi)}}\, f(\xi) d\xi$

$= \lambda \int_a^b \bar{k}(x,\xi) V(\xi)\, d\xi - \int_a^b \bar{k}(x,\xi) \frac{f(\xi)}{\sqrt{r(\xi)}}\, d\xi$ Where $\bar{k}(x,\xi) = G(x,\xi)\sqrt{r(x)}\,\sqrt{r(\xi)}$

If $G(x,\xi)$ is symmetric, then $\bar{k}(x,\xi)$ is also symmetric.

7. Find the integral eqn for the differential eqn Ly = y'' with boundary conditions y(0)= y(l) = 0

Solution: Ly = 0 $\Rightarrow$ y'' = 0, ie) $\frac{d^2y}{dx^2} = 0$

Integrating , $\frac{dy}{dx} = c_1 \Rightarrow dy = c_1 dx$

Again integrating $(0, l)$ $\int_0^l dy = \int_0^l c_1 dx$ ….. (1)

$[y]_0^l = c_1[x]_0^l + c_2 \Rightarrow y(l) - y(0) = c_1(l-0) + c_2 \Rightarrow 0 = c_1 l + c_2$

$\int_0^l dy = \int_0^x c_1 dx + \int_x^l c_1 dx$

$[y]_0^l = c_1[x]_0^x + c_1[x]_x^l \Rightarrow 0 = c_1 x + c_1(l-x)$ ….. (2)

$G(x,\xi) = \begin{Bmatrix} c_1 v(x), x < \xi \\ c_2 v(x), x > \xi \end{Bmatrix}$ where $c_1 = -\frac{v(\xi)}{A}, c_2 = -\frac{u(\xi)}{A}$

$$G(x,\xi) = \begin{Bmatrix} -\frac{v(\xi)}{A} u(x), x < \xi \\ -\frac{u(\xi)}{A} v(x), x > \xi \end{Bmatrix}$$

Here u(x) = x, v(x) = l – x (from (1))

By abel's formula, $u(x)v'(x) - v(x)u'(x) = \frac{A}{p(x)} \ldots\ldots (3)$

$u'(x) = 1, v'(x) = -1, p(x) = -1$

$[since\ Ly = y'' \Rightarrow py'' + p'y' + qy = y'' equating\ the\ coeff\ p = 1]$

$u(\xi)v'(\xi) - v(\xi)u'(\xi) = \frac{A}{p(\xi)}$

$\Rightarrow \xi(-1) - (l - \xi) = \frac{A}{1}$

$\Rightarrow -\xi + (-l + \xi) = A$

$\Rightarrow A = -l$

$$G(x,\xi) = \begin{Bmatrix} -\frac{(l-\xi)x}{-l}, x < \xi \\ -\frac{(-\xi(l-x))}{-l}, x > \xi \end{Bmatrix} = \begin{Bmatrix} -\frac{x(l-\xi)}{l}, x < \xi \\ -\frac{(\xi(l-x))}{l}, x > \xi \end{Bmatrix}$$

Therefore y(x) = $\int_0^l G(x,\xi)\, \phi(\xi) d\xi$

3. Solve the differential equation $y'' + \lambda r y = f(x)$ with boundary condition y(0)= y(l) = 0

Soln: $y'' + \lambda r y = f(x)$

$\Rightarrow y'' = f(x) - \lambda r(x) y(x)$

Integrating, $[y']_0^x = \int_0^x f(x_1)dx_1 - \lambda \int_0^x r(x_1)y(x_1)dx_1$

$y'(x) = \int_0^x f(x_1)dx_1 - \lambda \int_0^x r(x_1)y(x_1)dx_1 + y'(0)$

Again integrating, $[y(x)]_0^x = \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2 - \lambda \int_0^x \int_0^{x_2} r(x_1)y(x_1)dx_1dx_2 + y'(0)\int_0^x dx$

$\Rightarrow y(x) - y(0) = \int_0^x (x - \xi)f(\xi)d\xi - \lambda \int_0^x (x-\xi)r(\xi)y(\xi)d\xi + y'(0)x$

$\Rightarrow y(x) = \int_0^x (x - \xi)f(\xi)d\xi - \lambda \int_0^x (x-\xi)r(\xi)y(\xi)d\xi + y'(0)x$

Put x = l,

$\Rightarrow y(l) = \int_0^l (l-\xi) f(\xi) d\xi - \lambda \int_0^l (l-\xi) r(\xi) y(\xi) d\xi + y'(0) l$

$0 = \int_0^x (l-\xi) f(\xi) d\xi + \int_x^l (l-\xi) f(\xi) d\xi - \lambda \int_0^x (l-\xi) r(\xi) y(\xi) d\xi - \lambda \int_x^l (l-\xi) r(\xi) y(\xi) d\xi + y'(0) l$

$$y'(0) = \frac{1}{l}\left[\int_0^x (l-\xi) f(\xi) d\xi + \int_x^l (l-\xi) f(\xi) d\xi - \lambda \int_0^x (l-\xi) r(\xi) y(\xi) d\xi - \lambda \int_x^l (l-\xi) r(\xi) y(\xi) d\xi\right]$$

$= \frac{1}{l}\left[\int_0^x (l-\xi)[\lambda r(\xi) y(\xi) - f(\xi)] d\xi\right] + \int_x^l (l-\xi)[\lambda r(\xi) y(\xi) - f(\xi)] d\xi$

$y(x) = \int_0^x (x-\xi) f(\xi) d\xi - \lambda \int_0^x (x-\xi) r(\xi) y(\xi) d\xi + \frac{x}{l}\left[\int_0^x (l-\xi)[r(\xi) y(\xi) - f(\xi)] d\xi\right]$

$+ \frac{x}{l}\left[\int_x^l (l-\xi)[\lambda r(\xi) y(\xi) - f(\xi)] d\xi\right]$

$= \int_0^x \left[(x-\xi) - \frac{x}{l}((l-\xi))\right] f(\xi) d(\xi) + \int_0^x \frac{\lambda x}{l}(l-\xi) r(\xi) y(\xi) d\xi$

$+ \frac{x}{l}\left[\int_x^l (\xi - l)[f(\xi)] d\xi\right] + \frac{x}{l}\left[\int_x^l \frac{\lambda x}{l}(l-\xi)[r(\xi) y(\xi)] d\xi\right]$

$= \int_0^x \frac{\xi}{l}(x-l) f(\xi) d\xi + \int_0^x \frac{\lambda x}{l}(l-\xi) r(\xi) y(\xi) d\xi$

$+ \frac{x}{l}\left[\int_x^l (\xi - l)[f(\xi)] d\xi\right] + \frac{x}{l}\left[\int_x^l \frac{\lambda x}{l}(l-\xi)[r(\xi) y(\xi)] d\xi\right]$

$= \int_0^l G_1(x,\xi)\, \emptyset(\xi) d\xi + \int_0^l G_2(x,\xi)\, \emptyset(\xi) d\xi$

Where $G_1(x,\xi) = \begin{cases} \frac{\xi}{l}(l-x), \xi < x \\ \frac{x}{l}(\xi - l), \xi > x \end{cases}$; $G_2(x,\xi) = \begin{cases} \frac{\lambda\xi}{l}(l-\xi) r(\xi), \xi < x \\ \frac{\lambda x}{l}(l-\xi)[r(\xi)], \xi > x \end{cases}$

Alternative Definition of the Green's function.

4. Derive the integral eqn for non-homogenous end condition.

Soln: when the prescribed end conditions are not homogenous, we use the following method. Let $G(x,\xi)$ be the Green's function corresponding to the associated homogenous end condition. Let the integral eqn corresponding to the end eqn be y(x) = $\int_a^b G(x,\xi)\, \emptyset(\xi)\, d\xi$ …… (1)

Let us find the function $\emptyset(x)$ such that the relation y(x) = $\emptyset(x)$ + $\int_a^b G(x,\xi)\, \emptyset(\xi)\, d\xi$ …. (2) is equivalent to the differential eqn.

L(y(x)) + $\emptyset(x)$= 0 together with prescribed non – homogenous end condition.

Now, L(y(x)) = $-\,\emptyset(x) \Rightarrow$ L$\int_a^b G(x,\xi)\emptyset(\xi)\, d\xi = -\emptyset(x)$ ... (3)

(2) ⇒ y(x) = $p(x)$ + $\int_a^b G(x,\xi)\, \emptyset(\xi)\, d\xi$

L(y(x)) = L(p(x)) + L $\int_a^b G(x,\xi)\, \emptyset(\xi)\, d\xi$

$-\,\emptyset(x)$ = L(p(x)) + L $\int_a^b G(x,\xi)\emptyset(\xi)\, d\xi$

L(p(x)) = $-\emptyset(x)$ − L $\int_a^b G(x,\xi)\, \emptyset(\xi)\, d\xi$

0 = L(p(x))

Since y(x) = $\int_a^b G(x,\xi)\, \emptyset(\xi)\, d\xi$ satisfies the homogenous end condition if follows that the function p(x) in eqn (2) must be the solution of L(p(x)) = 0 which satisfies the non-homogenous end condition. Hence p(x) exists when $G(x,\xi)$ exists.

5. Solve the differential eqn $y'' + xy' = 1$ with the boundary condition y(0) = 0 and y(l) = 1

Soln: here the condition are non homogenous. Let $G(x,\xi) = \begin{Bmatrix} \frac{\xi}{l}(l-x), \xi < x \\ \frac{x}{l}(l-\xi), \xi > x \end{Bmatrix}$;

The Green's function corresponding to associated homogenous end condition y(0) = 0, y(l) = 0. We have to find p(x) = 0 such that p''(x) = 0 with p(0) = 0, p(l) = 1

Now p''(x) = 0 gives p'(x) = c_1, integrating we get p(x) = $c_1 x + c_2$

Put x = 0, p(0) = 0 + c_2 implies $c_2 = 0$

Put x = l, p(l) = $c_1 l + c_2$ implies $1 = c_1 l$ gives $c_1 = \frac{1}{l}$. Therefore $p(x) = \frac{x}{l}$

Now, the given is $y'' + xy' = 1$, ie) $\emptyset(x) = xy - 1$

The required solution is y(x) = $p(x) + \int_a^b G(x,\xi)\,\emptyset(\xi)\,d\xi$

y(x) = $\frac{x}{l} + \int_0^l G(x,\xi)\,\emptyset(\xi)\,d\xi$

$= \frac{x}{l} + \int_0^x \frac{\xi}{l}(l - x)(\xi\,y(\xi) - 1)\,d\xi + \int_x^l \frac{x}{l}(l - \xi)(\xi\,y(\xi) - 1)\,d\xi$

Type equation here. $= \frac{x}{l} + \frac{l-x}{l}\int_0^x (\xi^2\,y(\xi) - 1)\,d\xi + \frac{x}{l}\int_x^l \left(l\left(\xi\,y(\xi)\right) - l - \xi^2\,y(\xi) + \xi\right)d\xi$

$= \frac{x}{l} + \frac{l-x}{l}\int_0^x \left(\xi^2\,y(\xi)\right)d\xi - \frac{l-x}{l}\int_0^x (\xi)\,d\xi + \frac{x}{l}\int_x^l l\left(\xi\,y(\xi)\right)d\xi - \frac{x}{l}\int_x^l l\,d\xi - \frac{x}{l}\int_x^l y(\xi)\,\xi^2 d\xi + \frac{x}{l}\int_x^l \xi\,d\xi$

$= \frac{x}{l} + \frac{l-x}{l}\int_0^x \left(\xi^2\,y(\xi)\right)d\xi - \frac{l-x}{l}(\xi^2)_0^x + x\int_x^l \left(\xi \quad y(\xi)\right)d\xi - x(\xi\ )_0^x - \frac{x}{l}\int_x^l y(\xi)\,\xi^2 d\xi + \frac{x}{l}\left(\frac{\xi^2}{2}\right)_x^l$

$= \frac{x}{l} - \frac{l-x}{l}\left(\frac{x^2}{2}\right) - x(l - x) + \frac{x}{2l}(l^2 - x^2) + \frac{l-x}{l}\int_0^x (\xi^2\,y(\xi)d\xi)$

$+ x\int_x^l \left(\xi \quad y(\xi)\right)d\xi - \frac{x}{l}\int_x^l y(\xi)\,\xi^2 d\xi$

$= \frac{x}{l} - \frac{lx^2}{2l} + \frac{x^3}{2l} - xl + x^2 + \frac{xl^2}{2l} - \frac{x^3}{2l} + \frac{l-x}{l}\int_0^x (\xi^2\,y(\xi)d\xi) + \int_x^l (lx\xi - x\xi^2)\frac{y(\xi)}{l}\,d\xi$

$= \frac{x}{l} - \frac{xl}{2} + \frac{x^2}{2} + \frac{l-x}{l}\int_0^x (\xi^2\,y(\xi)d\xi) + \int_x^l (l\xi - \xi^2)x\frac{y(\xi)}{l}\,d\xi$

$$therefore\ G(x,\xi) = \begin{cases} \frac{\xi^2}{l}(l - x), \xi < x \\ \frac{x}{l}(l\xi - \xi^2), \xi > x \end{cases}$$

Hence $G(x,\xi)$ is not symmetric.

Bessel's function

Consider the eqn $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda x^2 - 1)y = 0$ with the boundary condition y(0) = y(1) = 0. Find the integral eqn corresponding to this equation.

Proof: here the given differential eqn involve λ. We compare the given eqn with

L(y(x)) + λr(x)y(x) = f(x) …… (1)

The given equation does not contain any x, therefore f(x) = 0

Then (1) ⇒ $p\,\frac{d^2y}{dx^2} + \frac{dp}{dx}\frac{dy}{dx} + qy + \lambda r(x)y(x) = f(x) \ldots.(2)$

We get, f(x) = 0, r(x) = x^2

If we take p = x^2 then it does not agree with the co-efficient of $\frac{dy}{dx}$

So let us choose p = x, so that $\frac{dp}{dx} = 1$ which satisfy the equation.

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \lambda xy - \frac{y}{x} = 0$$

Comparing this eqn with eqn (1), we get

p(x) = x, r(x) = x, q(x) = $-\frac{1}{x}$

Let us find the solution L(y) = 0

$$p\,\frac{d^2y}{dx^2} + \frac{dp}{dx}\frac{dy}{dx} + qy = 0$$

$$\Rightarrow x\,\frac{d^2y}{dx^2} + \frac{dy}{dx} + -\frac{1}{x}y = 0 \;\Rightarrow x^2\,\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

$$\Rightarrow (x^2D^2 + xD - 1)y = 0$$

Now, $xD = \theta; x^2D^2 = \theta(\theta - 1); x = e^{\theta}\,; logx = \theta$

Therefore $(\theta(\theta - 1) + \theta - 1)$y = 0

$$\Rightarrow (\theta^2 - \theta + \theta - 1)y = 0$$

$$\Rightarrow (\theta^2 - 1)y = 0$$

Auxillary eqn is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = 1$

$y = Ae^{-\theta} + Be^{\theta} = A(x-1) + Bx = \frac{A}{x} + Bx$

Here y is expressed as the linear transformation of x and x^{-1}. Hence all the linear transformation of x and x^{-1} are solutions of y.

Take two solutions u(x) and v(x) with the boundary condition u(0) = 0, u(1) = 0

Here we choose u(x) = x, v(x) = $\frac{1}{x} - x$,

W(u, v) = $uv' - vu'$

$$= x\left(-\frac{1}{x^2} - 1\right) - \left(\frac{1}{x} - x\right) = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \ldots..(3)$$

By Abel's formula,

$$uv' - vu' = \frac{A}{p(x)} \ldots..(4)$$

comparing (3)*and* (4), *we get* $A = -2$

Now, the general Green's function given by

$$G(x,\xi) = \begin{cases} -\frac{1}{A} u(x)v(\xi), x < \xi \\ -\frac{1}{A} v(x)u(\xi), x > \xi \end{cases} = \begin{cases} -\frac{1}{2} x\left(\frac{1}{\xi} - \xi\right), x < \xi \\ -\frac{1}{2}\left(\frac{1}{x} - x\right)\xi, x > \xi \end{cases}$$

$$G(x,\xi) = \begin{cases} \frac{1}{x}\left(\frac{1-\xi^2}{\xi}\right) x, x < \xi \\ \frac{1}{2}\left(\frac{1-x^2}{x}\right)\xi, x > \xi \end{cases}$$

Hence the corresponding integral eqn is

$$y(x) = \lambda \int_0^1 G(x,\xi)\xi\, y(\xi)d\xi$$

Unit 4

Linear equation in cause and effect

(The influence function)

1. Derive the influence function.

Physical problem by applying several causes, super imposing effect will be produced and thus linear integral equation arises.

Let x, ξ are variables which take the value in a certain common interval or a region R. x and ξ may represent the position.

Over the region R, let us suppose that the distribution of cause is acting active.

Our aim is to study the resultant distribution of effect in R denote the effect at x due to a unit cause concentrated at ξ by G(x, ξ). The differential effect at x due to a uniform distribution of cause of intensity C(ξ) over a elementary region (ξ, ξ + dξ) is given by C(ξ) G(x, ξ)dξ. Hence the effect R(x) at x due to a distribution of cause C(ξ) over the entire region R is given by

e(x) = $\int_R C(\xi)G(x,\xi)d\xi \ldots\ldots (1)$ if the effect due to the sum of two separate causes is the sum of the effect due to each of the causes.

The function G(x, ξ) which represents the effect at x due to a unit concentrated cause at ξ is often known as the influence function.

This function is either identical (or) proportional to the Green's function. If the distribution of cause is prescribed and if the influence function is known, then by using eqn (1) we can find the effect by direct integration.

If the problem is to determine the distribution of cause which will produce a known (or) desired effect distribution eqn (1) represents a Fredholm integral eqn of the first kind for determination of C.

Hence the kernel is identified by the influence function. If the cause and effect are not provided separately , that is, if they satisfying the linear relation of the form $C(x) = \emptyset(x) +$

$\lambda\, e(x) \ldots (2)$ where $\emptyset$ is the given function or 0 and λ is the constant, then the f and e can be eliminated between (1) and (2) we get

$$C(x) = \emptyset(x) + \int_R C(\xi)G(x,\xi)d\xi \ldots\ldots (3)$$

Which is the fred holm eqn of second kind, for the determination od ξcause distribution. If the cause C is eliminated between (1) and (2), we get

$$e(x) = \int_R G(x,\xi)\emptyset(\xi)d\xi + \lambda \int_R G(x,\xi)e(\xi)d\xi \ldots\ldots (4)$$

By using this we can determine the effect distribution. Both cause and effect are determined by solving either eqn (4) or (3) and using (2).

Linear eqn in Cause and Effect (the influence function)

In physical problem by applying several causes superimposing effect will be produced and thus linear and integral eqn arises.

Let x and ξ are variables which take the value in on certain common interval (or) a region R. X and ξ may represent a position

(in space of 1, 2 (or) 3 dimension) (or) time

Over the region R, let us suppose that a distribution of carve is acting active.

Therefore our aim is to study the resultant distribution of effect in R denote the effect at x due to a unit carve concentrated at ξ by G(x, ξ)

The differential effect at x due to a uniform distribution of carve of intensity c(ξ) over a elementary region (ξ, ξ+ dξ) is given by c(ξ) G(x, ξ)dξ

Hence the effect e(x) at x due to a distribution of cause c(ξ) over the entire region R is given by $e(x) = \int_R G(x,\xi)c(\xi)d\xi \ldots.$ (1) if superposition is valid, [ie) if the effect due to the sum of two separate causes is the sum of the effects due to each of the causes].

The function G(x, ξ) which represents the effect at x due to a unit concentrated cause at ξ is often known as the influence function.

The function is either identical or proportions to the Green's function. If the distribution of cause is prescribed and if the influence function is known, then by using eqn (1) we can find the effect by direct integration

If the problem is to determine the distribution of causes which will produce a known or desired effect distribution. Equation (1) represents a fredholm integral eqn of the first kind for a determination of c.

Hence the kernel is identified by the influence function.

If the cause and effect are not provided seperately (ie) If they satisfying the linear relation of the form

$$c(x) = \emptyset(x) + \lambda e(x) \ldots..(2)$$

where $\emptyset$ is a given function or zero and λ is a constant then the effect 'e' can be eliminated between (1) and (2), we get

c(x) = ∅(x) + λ$\int_a^b G(x,\xi)c(\xi)d\xi \ldots..(3)$ which is the fredholm eqn of 2nd kind for the determination of cause distribution.

If the cause c is eliminated between (1) and (2) we get,

e(x) = $\int_R G(x,\xi)\emptyset((\xi)d\xi + \lambda\int_R G(x,\xi)e(\xi)d\xi \ldots..(4)$

by using this, we can determine the effect distribution. Both cause and effect determined by solving either (4) or (3) and using (2).

1. Obtain the resolvent Kernel associated with $k(x,\xi) = e^{-(x-\xi)}$ in the interval (0, ∝)

Solution:

Here $k(x,\xi) = e^{-(x-\xi)} = k_1(x,\xi)$

$$k_2(x,\xi) = \int_0^{\propto} k_1(x,\xi_1)k_1(\xi_1,\xi)\,d\xi_1$$

$$= \int_0^{\propto} e^{-(x-\xi_1)} e^{-(\xi_1-\xi)} d\xi_1$$

$$= \int_0^{\propto} e^{-x+\xi_1-\xi_1+\xi}\, d\xi_1$$

$$k_2(x,\xi) = \int_0^{\propto} e^{-x+\xi}\, d\xi_1$$

$$= e^{-(x-\xi)}[\xi_1]_0^{\propto}$$

$$= e^{-(x-\xi)} \propto$$

$$k_3(x,\xi) = \int_0^{\propto} k_1(x,\xi_1)k_2(\xi_1,\xi)\,d\xi_1$$

$$= \int_0^{\propto} e^{-(x-\xi_1)} \propto e^{-}(\xi_1 - \xi\ )d\xi_1$$

$$= \propto \int_0^{\propto} e^{-x+\xi}\, d\xi_1$$

$$k_3(x,\xi) = \propto^2\, e^{-(x-\xi)}$$

$$k_4(x,\xi) = \propto^3\, e^{-(x-\xi)}$$

Similarly, $k_n(x,\xi) = \propto^{n-1}\, e^{-(x-\xi)}$

Fredholm eqn, $\Gamma x,\xi,\lambda = k(x,\xi) + \lambda \sum_{n=0}^{\infty} \lambda^n\, k_{n+2}(x,\xi)$

$$= e^{-x+\xi} + \lambda \sum_{n=0}^{\infty} \lambda^n \propto^{n+1} e^{-(x-\xi)}$$

$$= e^{-x+\xi}\,[1 + \lambda \propto + \lambda \propto^2 + \ \ldots]$$

$$= \frac{e^{-x+\xi}}{1-\lambda\propto}$$

2. Solve $y(x) = x + \lambda \int_0^1 (1 - 3xy)y(\xi)\,d\xi$

solution: $y(x) = x + \lambda \int_0^1 (1-3xy)y(\xi)\, d\xi$

Define a integral eqn,

$$kf(x) = \int_a^b k(x,\xi)f(\xi)\, d\xi$$

$$= \int_0^1 (1-3xy)f(\xi)\, d\xi$$

$$= \int_0^1 (1-3x\xi)(\xi)\, d\xi$$

$$= \int_0^1 (\xi - 3x\xi^2)\, d\xi = \left[\frac{\xi^2}{2} - \frac{3x\xi^2}{3}\right]_0^1$$

$$kf(x) = (\tfrac{1}{2} - \text{x}\,)$$

$$k^2f(x) = \int_0^1 kf(x)(1-3xy)\, d\xi$$

$$= \int_0^1 (1-3x\xi)\left(\frac{1}{2} - x\right) d\xi$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{3x\xi}{2} - x + 3x^2\xi\right) d\xi$$

$$= \left[\frac{\xi}{2} - \frac{3x\xi^2}{4} - x\xi + \frac{3x^2\xi^2}{2}\right]_0^1$$

$$= \left[\frac{1}{2} - \frac{3x}{4} - x + \frac{3x^2}{2}\right]$$

$$= [\frac{1}{2} - \frac{7x}{4} + \frac{3x^2}{2}]$$

$$k^3f(x) = \int_0^1 k^2f(x)(1-3x\xi)\, d\xi$$

$$= \int_0^1 (1-3x\xi)(\frac{1}{2} - \frac{7x}{4} + \frac{3x^2}{2})\, d\xi$$

$$= \int_0^1 (\frac{1}{2} - \frac{3x\xi}{2} - \frac{7x}{4} + \frac{21x^2\xi}{4} + \frac{3x^2}{2} - \frac{9x^3\xi}{2})d\xi$$

$$= \left[\frac{1}{2}\,\xi - \frac{3x}{4}\,\xi^2 - \frac{7x}{4}\,\xi + \frac{21x^2\xi^2}{(4)(2)} + \frac{3x^2}{2}\xi - \frac{9x^3}{(2)(2)}\xi^0\right]_0^1$$

$$= \frac{1}{2} - \frac{3x}{4} - \frac{7x}{4} + \frac{21x^2}{8} + \frac{3x^2}{2} - \frac{9x^3}{4}$$

$= \frac{1}{2} - \frac{5x}{2} + \frac{33x^2}{8} - \frac{9x^3}{4}$

Therefore y(x) = F(x) + λkF(x) + $\lambda^2 k^2 F(x) +$ ….

$$= x + \lambda\left(\frac{1}{2} - x\right) + \lambda^2\left(\frac{1}{2} - \frac{7x}{4} + \frac{3x^2}{4}\right) + \lambda^3\left(\frac{1}{2} - \frac{5x}{2} + \frac{33x^2}{8} - \frac{9x^3}{4}\right) + \ldots.$$

FREDHOLM EQUATION WITH SEPERABLE KERNELS

The kernel k(x, ξ) is seperable if it can be expressed as a sum of finite number of terms each of which is the product of function x alone and the function of ξ alone. It is expressed in the form

$k(x,\xi) = \sum_{n=1}^{N} f_n(x) g_n(\xi)$

Without loss of generality, we assume that the function $f_n(x)$ where n = 1,2,..N are linearly independent in the given interval.

Let $k(x,\xi) = \sin(x + \xi)$

$= sinx\ cos\xi + \cos x \sin \xi$

$= f_1(x)g_1(\xi) + f_2(x)g_2(\xi)$

$= \sum_{i=1}^{n} f_i(x) g_i(\xi)$

Therefore k(x,ξ) is seperable.

Note: Integral equations with separable kernel do not acquire frequently but we can easily solve the equation.

Theorem:

Solving the fred holm equation with separable kernel.

Proof: consider the fred holm eqn of second kind with separable kernel.

$$k(x,\xi) = \sum_{n=1}^{N} f_n(x) g_n(\xi)$$

The fred holm eqn is given by

y(x) = F(x) + $\lambda \int_a^b k(x,\xi)y(\xi)d\xi$ … (1)

= F(x) + $\lambda \int_a^b \sum_{n=1}^{N} f_n(x)g_n(\xi)\, y(\xi)d\xi$

y(x) = F(x) + $\lambda \int_a^b \sum_{n=1}^{N} f_n(x)g_n(\xi)\, y(\xi)d\xi$ …. (2)

The co-efficient of $f_1(x).f_2(x)$ ... $f_n(x)$ are considered to be the coefficient of c_1 , c_2 , ... c_N

$C_1 = \int_a^b g_1(\xi)\, y(\xi)d\xi$

$C_2 = \int_a^b g_2(\xi)\, y(\xi)d\xi \ldots .\ C_N = \int_a^b g_N(\xi)\, y(\xi)d\xi$

Hence the eqn becomes $y(x) = F(x) + \lambda \sum_{n=1}^{N} c_n f_n(x) \ldots .(3)$

which is the solution of integral eqn (1). The only thing which remains to find the constants c_1 , c_2 ... c_N. Hence the solution of Fredholm eqn can be approximated by the polynomial x and ξ.

Consider the eqn (3) we have $y(x) = F(x) + \lambda \sum_{n=1}^{N} c_n f_n(x)$

Multiply by $g_1(x)$,

$we\ get\ g_1(x)y(x) = g_1(x)F(x) + \lambda \sum_{n=1}^{N} c_n f_n(x)g_1(x)$

Integrating over a and b,

$$\int_a^b g_1(x)\, y(x)dx = \int_a^b g_1(x)\, F(x)dx + \lambda \int_a^b \sum_{n=1}^{N} c_n f_n(x)g_1(x)dx \ldots .(4)$$

Let $\alpha_{mn} = \int_a^b g_m(x)\, f_n(x)dx,\ \ \beta_m = \int_a^b g_m(x)\, F(x)dx$

Therefore $\int_a^b g_1(x)\, y(x)dx = \beta_1 + \lambda \sum_{n=1}^{N} c_n \alpha_{1n}$

$\Rightarrow\ c_1 = \beta_1 + \lambda \sum_{n=1}^{N} c_n \alpha_{1n}$

Similarly, multiplying the eqn (3) by $g_2(x)$ and integrating a and b we get,

$c_2 = \beta_2 + \lambda \sum_{n=1}^{N} c_n \alpha_{2n}$

$c_N = \beta_N + \lambda \sum_{n=1}^{N} c_n \alpha_{Nn}$. These c's eqn are (I)

Extending eqn (I), we get

$c_1(1 - \lambda\alpha_{11}) - c_2\alpha_{12}\,\lambda \ldots\ldots\ldots. -c_N\alpha_{1N}\,\lambda = \beta_1$

$- c_1\alpha_{21}\,\lambda + c_2(1 - \alpha_{22}\,\lambda) \ldots\ldots\ldots. - c_N\alpha_{2N}\,\lambda = \beta_2 \ldots..$

$- c_1\alpha_{N1} - \lambda c_2\alpha_{N2} \ldots\ldots.. - c_N(1 - \alpha_{NN}\,\lambda) = \beta_N \ldots\ldots\ldots\ldots (II)$

Hence we get N equations. This set of N eqns have a aunique solution for the c's if and only if the determiner Δ of the coefficient of $c_1, c_2 \ldots c_N$ is not equal to zero.

The above eqn can be written as (I - λA)C = B, where I is the unit matrix of order N and A is the matrix given by

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \ldots\ldots & \alpha_{1N} \\ \alpha_{21} & \alpha_{22} \ldots\ldots & \alpha_{2N} \\ \ldots\ldots.\alpha_{N1} & \alpha_{N1} \ldots\ldots & \alpha_{NN} \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} \; and\; C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}$$

This set of linear equation can be solved.

Case (1): If F(x) = 0 in (1), then we get $\beta_1 = 0, \beta_2 = 0, \ldots. \beta_N = 0$ hence the set of N eqns in (II) becomes a homogenous eqns.

Hence the trivial solution of (II) is $c_1 = c_2 = c_3 = \ldots. c_N = 0$ and the homogeneous eqn is satisfied by the trivial solution y(x) = 0

If $\Delta = |I - \lambda A| \neq 0$ then this is the only solution.

If $\Delta = 0$ then one of $c_1, c_2, \ldots. c_N$ can be assigned arbitrarily and remaining c's can be determined easily. In this case infinitely many solution of the integral eqn (1) we get the value of λ for which $\Delta(\lambda) = 0$ are called the characteristic values (or) eigen values.

Any non – trivial solution of the homogeneous integral equation is called the corresponding characteristic function.

If K of the constants $c_1, c_2, \ldots. c_N$ can be assigned arbitrarily for a given characteristic values of λ, then K linearly independent functions are obtained.

Case (2): If F(x) ≠ 0, then let it be orthogonal to all the functions $g_1(x), g_2(x) \ldots\ldots g_N(x)$ and we have $\beta_m = \int_a^b g_m(x)\, F(x) dx = 0$ for

m = 1,2,..

therefore the set of N eqns in (II) becomes

$c_1(1 - \lambda\alpha_{11}) - c_2\alpha_{12}\,\lambda \ldots\ldots\ldots. -c_N\alpha_{1N}\,\lambda = 0$

$-c_1\alpha_{21}\,\lambda + c_2(1 - \alpha_{22}\,\lambda) \ldots\ldots\ldots. - c_N\alpha_{2N}\,\lambda = 0 \ldots..$

$-c_1\alpha_{N1} - \lambda c_2\alpha_{N2} \ldots\ldots.. - c_N(1 - \alpha_{NN}\,\lambda) = 0 \ldots\ldots\ldots\ldots (II)$

Hence we can apply the previous discussion in this case also, but the solution of the integral eqn involves the function F(x). In this case the trivial solution is

$c_1 = 0, c_2 = 0, \ldots. c_N = 0$ but since F(x) ≠ 0 we get y(x) = F(x)

Note: the solution corresponding to the characteristic value λ is now expressed as a sum of F(x) and arbitrarily multiples of the characteristic functions. If atleast one of the right hand members of II does not vanish a unique non-trivial solution of II exists and hence there exist a non-trivial solution of the given integral eqn if λ ≠ 0

Example: Solve the integral eqn $y(x) = \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi + F(x)$

Solution: The Fredholm eqn is

$y(x) = \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi + F(x) \ldots..(1)$

$k(x,\xi) = 1 - 3x\xi = f_1(x)g_1(\xi) - f_2(x)g_2(\xi)$

where $f_1(x) = 1, g_1(\xi) = 1\; f_2(x) = 3x, g_2(\xi) = \xi$

therefore $k(x,\xi) = \sum_{n=1}^{N} f_n(x) g_n(\xi)$.

k(x, ξ) is separable.

$$y(x) = \lambda \int_0^1 y(\xi)d\xi - 3x\lambda \int_0^1 (\xi)y(\xi)d\xi + F(x)$$

$$c_1 = \int_0^1 y(\xi)d\xi, \quad c_2 = \int_0^1 (\xi)y(\xi)d\xi$$

y(x) = $\lambda[c_1 - 3xc_2] + F(x) \ldots.(2)$

multiply by $g_1(x)$

$y(x)g_1(x) = \lambda[c_1 - 3xc_2]g_1(x) + F(x)g_1(x)$

Integrating we get,

$\int_0^1 y(x)\, g_1(x)dx = \lambda \int_0^1 g_1(x)\, [c_1 - 3xc_2]dx + \int_0^1 F(x)\, g_1(x)dx$

$$\int_0^1 y(x)\, dx = \lambda \int_0^1 [c_1 - 3xc_2]\, dx + \int_0^1 F(x)\, dx$$

$$c_1 = \lambda \left[c_1 x - \frac{3x^2}{2} c_2\right]_0^1 + \int_0^1 F(x)\, dx$$

$$c_1 - c_1 \lambda + \frac{3}{2} c_2 \lambda = \int_0^1 F(x)\, dx$$

$$\int_0^1 F(x)\, dx = (1 - \lambda)c_1 + \frac{3}{2}\lambda c_2 \ldots..(3)$$

Multiply by $g_2(x)$ in eqn (2)

$y(x)g_2(x) = \lambda[c_1 - 3xc_2]g_2(x) + F(x)g_2(x)$

integrating we get,

$\int_0^1 y(x)\, g_2(x)dx = \lambda \int_0^1 [c_1 - 3xc_2]\, g_2(x)dx + \int_0^1 F(x)\, g_2(x)dx$

$\int_0^1 xy(x)\, dx = \lambda \int_0^1 [c_1 - 3xc_2]\, (x)dx + \int_0^1 F(x)\, (x)dx$

$$c_2 = \lambda\left[\frac{c_1x^2}{2} - \frac{3x^3}{3}c_2\right]_0^1 + \int_0^1 xF(x)\,dx$$

$$\Rightarrow c_2 = \frac{\lambda c_1}{2} - \lambda c_2 + \int_0^1 xF(x)\,dx$$

$$\Rightarrow c_2 + \frac{\lambda c_1}{2} - \lambda c_2 = \int_0^1 xF(x)\,dx$$

$$\Rightarrow c_2(1+\lambda) - \frac{\lambda c_1}{2} = \int_0^1 xF(x)\,dx \ldots.(4)$$

Since F(x) ≠ 0, let it be orthogonal to the function $g_1(x), g_2(x)$

Hence $\int_a^b g_m(x)F(x)dx = 0, \quad m = 1,2, \ldots.$

From (3) and (4), $(1-\lambda)c_1 + \frac{3\lambda c_2}{2} = 0;\ (1+\lambda)c_2 - \frac{\lambda c_1}{2} = 0 \ldots\ldots(I)$

Solving (I) Type equation here.

$$\Delta\lambda = \begin{vmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{\lambda}{2} & 1+\lambda \end{vmatrix}$$

$$= (1-\lambda^2) + \frac{3}{4}\lambda^2 = 1 - \frac{\lambda^2}{4}$$

Now (II) has unique soln if and only if $\Delta\lambda \neq 0$

$$\Rightarrow 1 - \frac{\lambda^2}{4} \neq 0$$

$$\Rightarrow \frac{\lambda^2}{4} \neq 1 \Rightarrow \lambda^2 \neq 4 \Rightarrow \lambda \neq \pm 2$$

The unique soln is obtained by $\lambda \neq \pm 2$. The unique soln is obtained for c_1 *and* c_2 and substitute the values of c_1 *and* c_2 in eqn (3)

Case (1): homogeneous case

If F(x) = 0, $\lambda \neq \pm 2$, the unique solution is only the trivial solution $c_1 = 0, c_2 = 0, y(x) = 0$

The numbers $\lambda \neq \pm 2$ are the characteristic numbers.

If $\lambda = 2$ eqn (3) becomes

(3) $\Rightarrow \int_0^1 F(x)\,dx = (1 - \ 2)c_1 + \frac{3}{2}2c_2$

$\int_0^1 F(x)\,dx = -c_1 + 3\lambda c_2$

(4) $\Rightarrow \Rightarrow c_2(1 + \ 2) - \frac{2c_1}{2} = \ \int_0^1 xF(x)\,dx$

$\Rightarrow 3c_2 - c_1 = \ \int_0^1 xF(x)\,dx$

Now , $\int_0^1 F(x)\,dx = -c_1 + 3c_2$

$\int_0^1 xF(x)\,dx = 3c_2 - c_1$. Consider these 2 eqns as (II)

If $\lambda = -2$,

$\int_0^1 F(x)\,dx = 3c_1 - 3c_2$

$\Rightarrow \frac{1}{3}\int_0^1 F(x)\,dx = c_1 - c_2$

$\int_0^1 xF(x)\,dx = -c_2 + c_1$. Consider these 2 eqns as (III)

Eqn (II) are incompatible unless the function F(x) satisfy the eqn then,

$\int_0^1 F(x)\,dx - \int_0^1 xF(x)\,dx = 0$

$\Rightarrow \int_0^1 (1 - x)F(x)\,dx = 0 \ldots\ldots (5)$

Eqn (III) are incompatible unless the function F(x) satisfy the eqn, then $\frac{1}{3}\int_0^1 F(x)\,dx - \int_0^1 xF(x)\,dx = 0$

$\Rightarrow \ \int_0^1 (\frac{1}{3} - x)F(x)\,dx = 0 \ldots. (6)$

And in this case the corresponding pair of eqns (II) and (III) are redundant.

Let F(x) = 0

Eqn (1) becomes homogeneous.

(II) $\Rightarrow -c_1 + \ 3c_2 = 0 \ \ \Rightarrow c_1 = 3c_2$

(III) $\Rightarrow c_1 - c_2 = 0 \quad \Rightarrow c_1 = c_2$

Here $\lambda = 2, c_1 = 3c_2$

(2) ⇒y(x) = $\lambda(c_1 - 3c_2x) + F(x)$

= $\lambda(3c_2 - 3c_2x) + F(x)$

= $\lambda 3c_2(1 - x) + 0$

= $6c_2(1 - x)$

y(x) = A(1 – x)

Hence the function (1 – x) is the characteristic function corresponding to λ = 2.

Here $\lambda = -2, c_1 = c_2$

(2) ⇒y(x) = $\lambda(c_1 - 3c_1x) + F(x)$

= -2$(c_1 - 3xc_1) + 0$

= - 2 $c_1(1 - 3x)$

y(x) = B(1 – 3x)

Hence the function (1 – 3x) is the characteristic function corresponding to

λ = -2 corresponding to the two characteristic function we can write,

$$y(x) = D_1(1 - x) + D_2(1 - 3x) + F(x) \ldots\ldots(7)$$

Comparing (2) and (7)

$\lambda[c_1 - 3xc_2] + F(x) = D_1(1 - x) + D_2(1 - 3x) + F(x)$

$\lambda[c_1 - 3xc_2] = (D_1 - D_1x) + D_2 - 3xD_2$

$\lambda c_1 - 3\lambda xc_2 = (D_1 + D_2) - (D_1 + 3D_2)x$

$D_1 + D_2 = \lambda c_1$

$D_1 + 3D_2 = 3\lambda c_2$

Solving these two eqns we get, $2D_2 = \lambda c_1 - 3\lambda c_2$

$D_2 = \frac{(3c_2 - c_1)\lambda}{2}$

$D_1 + \frac{(3c_2 - c_1)\lambda}{2} = \lambda c_1$

$D_1 = \frac{3(c_1 - c_2)\lambda}{2}$

Hence $y(x) = D_1(1-x) + D_2(1-3x) + F(x)$ is the solution of given eqn where $D_1 = \frac{3(c_1 - c_2)\lambda}{2}$, $D_2 = \frac{(3c_2 - c_1)\lambda}{2}$

Case 2: non-homogeneous

If F(x) $\neq 0$, the unique soln exists if $\lambda \neq \pm 2$

If λ =2, $\int_0^1 (1-x)F(x)\,dx = 0$ shows that no solution exists unless F(x) is orthogonal to the characteristic corresponding to λ = 2

Using (II) we have,

$-c_1 + 3c_2 = \int_0^1 F(x)\,dx \Rightarrow c_1 = 3c_2 - \int_0^1 F(x)\,dx$

(2) ⇒ y(x) = $\lambda(c_1 + 3c_2 x) + F(x)$

$$= \lambda\left[3c_2 - \int_0^1 F(x)\,dx - 3c_2 x\right] + F(x)$$

$$= \lambda\left[3c_2(1-x) - \int_0^1 F(x)\,dx\right] + F(x)$$

$$= \lambda\left[E(1-x) - \int_0^1 F(x)\,dx\right] + F(x)$$

$$= \left[2E(1-x) - 2\int_0^1 F(x)\,dx\right] + F(x)$$

Thus in case infinitely many solutions exist differing from each other by a multiple of relevant characteristic function.

Similarly, if λ = - 2, then there is no solution unless F(x) is orthogonal to (1- 3x) over the interval 0 to 1.

In this case also infinitely many solution exists as follows.

Using (III) we have,

$c_1 - c_2 = \frac{1}{3}\int_0^1 F(x)\,dx$

$c_1 = c_2 + \frac{1}{3}\int_0^1 F(x)\,dx$

(2) ⇒ y(x) = λ$(c_1 - 3c_2 x) + F(x)$

= -2 $[c_2 + \frac{1}{3}\int_0^1 F(x)\,dx - 3c_2 x] + F(x)$

$= -2c_2 - \frac{2}{3}\int_0^1 F(x)dx + 6c_2 x + F(x)$

$= -2c_2(1-3x) - \frac{2}{3}\int_0^1 F(x)dx + F(x)$

$=Z(1-3x) - \frac{2}{3}\int_0^1 F(x)dx + F(x)$

Fredholm theory:

Let the continuous kernel k(x, ξ) need not be real.

Γ(x, ξ) $= \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,\xi) = k(x,\xi) + \lambda k_2(x,\xi) + \lambda^2 k_3(x,\xi)$ is called the resolvent kernel.

It can be expressed as the ratio of 2 infinite converging power series.

Now $\Gamma(x,\xi,\lambda) = \frac{D(x,\xi,\lambda)}{\Delta\lambda} \ldots.(1)$ where $D(x,\xi,\lambda) = k(x,\xi) + \lambda D_1(x,\xi) + \lambda^2 D_2(x,\xi) + \ldots.\, and\ \Delta\lambda = 1 + \lambda c_1 + \lambda^2 c_2 + \ldots$

It is found that coefficient c_n and the function $D_n(x,\xi)$ can be determined successfully by the following sequence of calculations.

$c_1 = \int_a^b k(x,x)\,dx,\ 2c_2 = -\int_a^b D_1(x,x)\,dx\ , \ldots.\ nc_n = -\int_a^b D_{n-1}(x,x)\,dx$

Γ(x, $\xi,\lambda)\Delta\lambda = D(x,\xi,\lambda)$

$(k(x,\xi) + \lambda k_2(x,\xi) + \lambda^2 k_3(x,\xi) + \ldots)(1 + \lambda c_1 + \lambda^2 c_2 + \lambda^3 c_3 + \cdots)$

$\Rightarrow \quad k(x,\xi) + \lambda c_1 k(x,\xi) + \lambda^2 c_2 k\,(x,\xi), \ldots + (\lambda k_2(x,\xi) + \lambda^2 c_1 k_2(x,\xi) + \lambda^3 c_2 k_2(x,\xi) + (\lambda^2 k_3(x,\xi) + \lambda^3 c_1 k_3(x,\xi) + \lambda^4 c_2 k_3(x,\xi) + \ldots = k(x,\xi) + \lambda D_1(x,\xi) + \lambda^2 D_2(x,\xi)) + \ldots$

$D_1(x,\xi) = c_1 k(x,\xi) + k_2(x,\xi)$

$D_1(x,\xi) = c_1 k(x,\xi) + \int_a^b k(\xi_n,\xi)\, k(x,\xi) d\xi$ .... (2)

$D_2(x,\xi) = c_2 k(x,\xi) + c_1 k_2(x,\xi) + k_3(x,\xi)$

$$= c_2 k(x,\xi) + c_1 \int_a^b k(x,\xi) k(\xi_1,\xi) d\xi_1 + \int_a^b k(x,\xi_1) k_2(\xi_1,\xi) d\xi_1$$

$$= c_2 k(x,\xi) + \int_a^b k(x,\xi_1)\,[c_1 k(\xi_1,\xi) + k_2(\xi_1,\xi)] d\xi_1$$

$$D_2(x,\xi) = c_2 k(x,\xi) + \int_a^b k(x,\xi_1)\, D_1(\xi_1,\xi) d\xi_1$$

⋮

$$D_n(x,\xi) = c_n k(x,\xi) + \int_a^b k(x,\xi_1)\, D_{n-1}(\xi_1,\xi)\, d\xi_1$$

To find the soln of the eqn

y(x) = F(x) + $\lambda \int_a^b k(x,\xi) y(\xi) d\xi$

the soln in terms of the resolvent kernel is given by

$y(x) = F(x) + \lambda \int_a^b \Gamma(x,\xi,\lambda) F(\xi) d\xi$

$$\Rightarrow\; y(x) = F(x) + \lambda \int_a^b \frac{D(x,\xi,\lambda)}{\Delta\lambda} F(\xi) d\xi \;\ldots.(3)$$

In the above eqn if $k(x,\xi)$ is separable, then the result is equivalent to the solution obtained by the method in the sec 3.6.

If this ratio is expanded as single power series in λ the result must be of the form,

$$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n k^n F(x)$$

This will converge for small value of $|\lambda|$, when $|\lambda| < |\lambda_1|$

The numerator series of the denominator series in eqn (3) will converge for all values of λ. If λ takes the characteristic value then the denominator value $\Delta\lambda$ vanishes. In this case no solution (or) infinitely many soln of the eqn

$y(x) = F(x) + \lambda\int_a^b k(x,\xi)y(\xi)d\xi$ exists.

Theorem:

In the following paragraph we summarized the certain known facts which generalize results already obtained in the special cases ie) when the kernel is either separable real (or) symmetric..

The eqn $y(x) = F(x) + \lambda\int_a^b k(x,\xi)y(\xi)d\xi \ldots$ (1) where (a, b) is a fiite interval and where F(x) and k(x, ξ) are continuous in (a, b) possess one and only one continuous solution for any fixed value of λ which is not a characteristic value.

If λ_c is a characteristic number of multiplicity of r that is if the associated homogeneous eqn

$$y(x) = \lambda_c \int_a^b k(x,\xi)y(\xi)d\xi \ldots. (2)$$

Possess r linearly independent non-trivial solns, $\varphi_1, \varphi_2, \varphi_3, \ldots \varphi_r$, then r is finite and the associated transposed homogeneous eqn

$z(x) = \lambda_c \int_a^b k(x,\xi)z(\xi)d\xi \ldots.$ (3) also possess r linearly independent non-trivial solutions $\Psi_1, \Psi_2, \ldots. \Psi_r$. In this exceptional case (1) possess no soln unless F(x) is orthogonal to each of the characteristic function $\Psi_1, \Psi_2, \ldots. \Psi_r$,

$$\int_a^b F(x)\Psi_k(x)dx = 0 \quad (k = 1,2, \ldots r) \ldots \ldots (3)$$

Finally if $\lambda = \lambda_c$ and (4) is satisfied then the solution of (1) is determinate only within an additive linear combination G.

$c_1\varphi_1 + c_2\varphi_2 + \cdots c_r\varphi_r$ where r constants, c_n are arbitrary. If k(x, ξ) is real and symmetric the eqn (2) and (3) are identical and the preceeding results reduces to the Hilbert's Schmidt theory.

Note: The volterra integral eqn y(x) = F(x) + λ$\int_a^b k(x,\xi)y(\xi)d\xi$ can be consider as the special form of Fredholm eqn with a kernel given by

$$\tilde{k}(x,\xi) = \begin{Bmatrix} 0 \; if \; x < \xi \\ k(x,\xi) \; if \; x > \xi \end{Bmatrix}$$

Unless k(x,x) = 0 the modified kernel $\tilde{k}(x,\xi)$ is discontinuous when x = ξ

Therefore if F(x) and $k(x,\xi)$ are continuous, the volterra eqn

y(x) = F(x) + λ$\int_a^b k(x,\xi)y(\xi)d\xi$ possess only one continuous solution and that solution is given by $y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n (k_x)^n F(x)$ for any value of λ.

If F(x) = 0, then it has only one trivial continuous solution y(x) = 0

A) Obtain the resolvent kernel associated with $k(x,\xi) = e^{-(x-\xi)}$ in interval (0, α)

Soln: here $k(x,\xi) = e^{-(x-\xi)} = k_1(x,\xi)$

$$k_2(x,\xi) = \int_0^{\alpha} k_1(x,\xi)k_1(\xi,\xi)d\xi_1$$

$$= \int_0^{\alpha} e^{-(x-\xi_1)}e^{-((\xi)-\xi)}\, d(\xi_1)$$

$$= \int_0^{\alpha} e^{-x+\xi_1-\xi_1+\xi} d\xi_1$$

$$k_2(x,\xi) = e^{-(x-\xi)} \int_0^{\alpha} d\xi_1 \;\; = e^{-(x-\xi)}[\alpha]$$

$$k_3(x,\xi) = \int_0^{\alpha} k_1(x,\xi)k_2(\xi,\xi)d\xi_1$$

$$= \int_0^{\alpha} e^{-(x-\xi_1)}e^{-((\xi)-\xi)}\alpha\, d(\xi_1)$$

$$= \alpha \int_0^{\alpha} e^{-x+\xi_1-\xi_1+\xi} d\xi_1$$

$$= \alpha e^{-(x-\xi)} \int_0^{\alpha} d\xi_1$$

$$k_3(x,\xi) = e^{-(x-\xi)}[\alpha^2]$$

$$k_n(x,\xi) = e^{-(x-\xi)}[\alpha^{n-1}]$$

$\Gamma(x,\xi,\lambda) = k(x,\xi) + \lambda \sum_{n=0}^{\infty} \lambda^n k_{n+2}(x,\xi)$

$$= e^{-(x-\xi)} + \lambda \sum_{n=0}^{\infty} \lambda^n e^{-(x-\xi)}[\alpha^{n+1}]$$

$$= e^{-(x-\xi)} + \left[1 + \lambda \sum_{n=0}^{\infty} \lambda^n [\alpha^{n+1}]\right]$$

$$= e^{-(x-\xi)}[1 + \lambda\alpha + \lambda^2\alpha^2 + \ldots]$$

$$= e^{-(x-\xi)}\,[1-(\lambda\alpha)]^{-1}$$

$$= \frac{e^{-(x-\xi)}}{[1-[\lambda\alpha]]}$$

Iterative method:

Problem: 1

y(x) = x + $\lambda \int_0^1 (1-3x\xi)y(\xi)d\xi$

soln:

Define an integral operator

kf(x) = $\int_a^b k(x,\xi)f(\xi)d\xi = \int_a^b (1-3x\,\xi)f(\xi)d\xi$

The solution of the fredholm eqn by using iterative method.

$$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n (\,k\,\,)^n F(x)$$

Now, $kf(x) = \int_0^1 (1-3x\,\xi)f(\xi)d\xi$

$= \int_0^1 (1-3x\,\xi)(\xi)d\xi$

= $\int_a^b (\xi - 3x\,\xi^2)d\xi$

= $[\frac{\xi^2}{2} - \frac{3x\xi^3}{3}]_0^1$

kf(x) = (1/2 - x)

Now, $k^2 f(x) = \int_0^1 (1-3x\,\xi)kf(\xi)d\xi$

$= \int_0^1 (1-3x\,\xi)\left(\frac{1}{2} - \xi\right)d\xi$

$= \int_0^1 \left(\frac{1}{2} - \xi - 3x\,\xi + 3x\xi^2\right)d\xi$

= $[\frac{\xi}{2} - \frac{\xi^2}{2} - \frac{3x\xi^2}{2} + \frac{3x\xi^3}{3}]_0^1$

= $\frac{1}{2} - \frac{1}{2} - \frac{3x}{2} + x$

$k^2f(x) = -\frac{x}{2}$

Now, $k^3 f(x) = \int_0^1 (1-3x\,\xi)k^2 f(\xi)d\xi$

$= \int_0^1 (1-3x\,\xi)\left(-\frac{\xi}{2}\right)d\xi$

$= -\frac{1}{2}\int_0^1 (1-3x\,\xi)(\xi)d\xi$

$= -\frac{1}{2}(\frac{1}{2} - x)$

Now, $k^4 f(x) = \int_0^1 (1 - 3x\,\xi) k^3 f(\xi) d\xi$

$= \int_0^1 (1 - 3x\,\xi) - \frac{1}{2}\left(\frac{1}{2} - \xi\right) d\xi$

$= -\frac{1}{2}(-\frac{x}{2})$

$y(x) = F(x) + \lambda^1 kF(x) + \lambda^2 k^2 F(x) + \lambda^3 k^3 F(x) + \lambda^4 k^4 F(x) + \ldots$

$= x + \lambda^1(1/2 - x) + \lambda^2 \left(-\frac{x}{2}\right) + \lambda^3\left(-\frac{1}{2}\left(\frac{1}{2} - x\right)\right) + \lambda^4\left(-\frac{1}{2}\left(-\frac{x}{2}\right)\right) + \cdots$

$= (x - (\lambda^2 x)\frac{1}{2} + \frac{\lambda^4}{4}x - \cdots) + \lambda^1(\frac{1}{2} - x)(1 - \frac{\lambda^2}{2} + \frac{\lambda^4}{4} - \cdots)$

$= x(1 - (\lambda^2)\frac{1}{2} + \frac{\lambda^4}{4} - \ \ldots) + \lambda^1\left(\frac{1}{2} - x\right)(1 - \frac{\lambda^2}{2} + \frac{\lambda^4}{4} + \ \ldots)$

$y(x) = \left(1 - \frac{\lambda^2}{2} + \frac{\lambda^4}{4} - \ \ldots\right)\left(x + \lambda^1\left(\frac{1}{2} - x\right)\right)$

Problem 2:

y(x) = 1 + $4\int_0^1 (1 - 3x\xi) y(\xi) d\xi$

solution: define an integral operator,

kf(x) = $\int_a^b k(x,\xi) f(\xi) d\xi = \int_0^1 (1 - 3x\,\xi) f(\xi) d\xi$

$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n (k\ \ )^n F(x)$

$kf(x) = \int_0^1 (1 - 3x\,\xi) f(\xi) d\xi$

$= \int_0^1 (1 - 3x\,\xi) d\xi$

$= [\xi - 3x\frac{\xi^2}{2}]_0^1$

$kf(x) = 1 - \frac{3x}{2}$

Now, $k^2 f(x) = \int_0^1 (1 - 3x\,\xi) kF(\xi) d\xi$

$= \int_0^1 (1 - 3x\,\xi)(1 - \frac{3\xi}{2}) d\xi$

$= \int_0^1 \left(\frac{1}{2} - \frac{3\xi}{2} - 3x\,\xi + \frac{9x\xi^2}{2}\right) d\xi$

$= [\frac{\xi}{1} - \frac{3x\xi^2}{2} - \frac{3\xi^2}{4} + \frac{9x\xi^3}{6}]_0^1$

$= [1 - \frac{3x}{2} - \frac{3}{4} + \frac{3x}{2}]$

$k^2 F(x) = \frac{1}{4}$

Now, $k^3 f(x) = \int_0^1 (1 - 3x\,\xi) k^2 F(\xi) d\xi$

$= \int_0^1 (1 - 3x\,\xi) \left(\frac{1}{4}\right) d\xi$

$= \frac{1}{4} \int_0^1 (1 - 3x\,\xi) d\xi$

$k^3 f(x) = \frac{1}{4}\left(1 - \frac{3x}{2}\right)$

Now, $k^4 f(x) = \int_0^1 (1 - 3x\,\xi) k^3 F(\xi) d\xi$

$= \int_0^1 (1 - 3x\,\xi) \frac{1}{4}\left(1 - \frac{3\xi}{2}\right) d\xi$

$= \frac{1}{4} \int_0^1 (1 - 3x\,\xi) \left(1 - \frac{3\xi}{2}\right) d\xi$

$= \frac{1}{4} . \frac{1}{4}$

$k^4 f(x) = \frac{1}{16}$

therefore the solution is

y(x) = $F(x) + \lambda\, kF(x) + \lambda^2 k^2 F(x) + \lambda^3 k^3 F(x) + \ldots$

$= 1 + 4\left(1 - \frac{3x}{2}\right) + 4^2 . \frac{1}{4} + 4^3 . \frac{1}{4}\left(1 - \frac{3x}{2}\right) + 4^4 . \frac{1}{16} + \ldots$

$$= \left(1 + \frac{4^2}{4} + \frac{4^4}{16} + \ \ldots.\right) + 4\left(1 - \frac{3x}{2}\right)\left(1 + \frac{4^2}{4} + \frac{4^4}{16} + \cdots\right)$$

$$= \left(1 + \frac{4^2}{4} + \frac{4^4}{16} + \ \ldots\right)(1 + 4(1 - \frac{3x}{2}))$$

$$y(x) = (1 + 4 + 4^2 + \ \ldots)(1 + 4(1 - \frac{3x}{2}))$$

Unit- V

Hillbert Schmidt Theory:

When $k(x,\xi)$ is not separable where $k(x,\xi)$ is the kernel of the homogenous Fredholm equation, this theory can be used.

Let $k(x,\xi)$ be given by different analytic function in the interval $x < \xi$ and $x > \xi$.

There are different characteristics numbers λ_n $(n = 1, 2, \dots)$ and corresponding we get different characteristic functions within an arbitrary constants.

In some exceptional cases a characteristic number λ_k may correspond to two or more characteristic functions.

Proposition 1:

If $y_m(x)$ and $y_n(x)$ are characteristic functions of $y(x) = \lambda \int_a^b k(x,\xi)y(\xi)d\xi$ corresponding to two different characteristic numbers λ_m and λ_n respectively. Then $y_m(x)$ and $y_n(x)$ are orthogonal over the interval (a,b) if the kernel $k(x,\xi)$ is symmetric.

Proof:

Let $y(x) = \lambda \int_a^b k(x,\xi)y(\xi)d\xi$ __________(1) be the given Fredholm equation

Let λ_m and λ_n be the two different characteristic numbers and let $y_m(x)$ and $y_n(x)$ be the characteristic functions corresponding to the characteristic numbers λ_m and λ_n respectively.

Let $k(x,\xi)$ be symmetric then we have $k(x,\xi) = k(\xi,x)$

The functions $y_m(x)$ and $y_n(x)$ must satisfy the equation (1)

Hence we get,

$$y_m(x) = \lambda_m \int_a^b k(x,\xi)y_m(\xi)d\xi \quad \text{\_\_\_\_\_\_\_(2)}$$

$$y_n(x) = \lambda_n \int_a^b k(x,\xi)y_n(\xi)d\xi \quad \text{\_\_\_\_\_\_\_(3)}$$

Multiply (2) by $y_n(x)$ we get,

$$(2) \Rightarrow y_n(x)\, y_m(x) = \lambda_m \int_a^b k(x,\xi) y_m(\xi) d\xi\, y_n(x)$$

Integrate with respect to x

$$\int_a^b y_n(x)\, y_m(x)\, dx = \lambda_m \int_a^b y_m(\xi) \left[\int_a^b k(x,\xi)\, y_n(x) d\xi \right] dx$$

$$\int_a^b y_m(x)\, y_n(x)\, dx = \lambda_m \int_a^b y_m(\xi) \left[\int_a^b k(x,\xi)\, y_n(x) dx \right] d\xi \_\_\_\_\_(4)$$

Consider equation (3)

$$(3) \Rightarrow y_n(x) = \lambda_n \int_a^b k(x,\xi) y_n(\xi) d\xi$$

$$y_n(\xi) = \lambda_n \int_a^b k(\xi,x) y_n(x) dx$$

$$\Rightarrow y_n(\xi) = \lambda_n \int_a^b k(x,\xi) y_n(x) dx \_\_\_\_\_(5)$$

Substitute (5) in (4)

$$\int_a^b y_m(x)\, y_n(x)\, dx = \lambda_m \int_a^b y_m(\xi)\, \frac{y_n(\xi)}{\lambda_n}\, d\xi$$

$$\lambda_n \int_a^b y_m(x)\, y_n(x)\, dx = \lambda_m \int_a^b y_m(\xi)\, y_n(\xi) d\xi$$

$$\lambda_n \int_a^b y_m(x)\, y_n(x)\, dx = \lambda_m \int_a^b y_m(x)\, y_n(x) dx$$

$$(\lambda_n - \lambda_m) \int_a^b y_m(x)\, y_n(x)\, dx = 0$$

Since $\lambda_n - \lambda_m \neq 0$

$$\Rightarrow \int_a^b y_m(x)\, y_n(x)\, dx = 0$$

Therefore, we conclude that the characteristic functions $y_m(x)\ y_n(x)$ are orthogonal to over the integral (a, b)

Note:

The above result is true only when the kernel is symmetric

Proposition 2:

The characteristic numbers of a Fredholm equation with real symmetric kernel are all real

Proof:

Let λ_m be a characteristic number of a fredholm equation with real symmetric kernel

To prove: λ_m is real

If suppose λ_m is not real then $\lambda_m = \alpha_m + i\beta_m$

Let the corresponding characteristic function be complex given by

$$y_m(x) = f_m(x) + ig_m(x)$$

Let $\overline{y_m(x)}$ be the complex conjugate of $y_m(x)$, then $\overline{\lambda_m}$ the complex conjugate of λ_m is the characteristic number corresponding to the function $\overline{y_m(x)}$.

From the previous proposition we have,

$$(\lambda_m - \lambda_n) \int_a^b y_m(x)\, y_n(x)\, dx = 0$$

Replace λ_n by $\overline{\lambda_m}$ and $y_n(x)$ by $\overline{y_m(x)}$, we have

$$(\lambda_m - \overline{\lambda_m}) \int_a^b y_m(x)\, \overline{y_m(x)}\, dx = 0 \qquad \text{\_\_\_\_\_\_\_\_}(1)$$

$$\lambda_m - \overline{\lambda_m} = \alpha_m + i\beta_m - \left(\overline{\alpha_m + \imath\beta_m}\right)$$

$$= \alpha_m + i\beta_m - (\alpha_m - i\beta_m)$$

$$= \alpha_m + i\beta_m - \alpha_m + i\beta_m$$

$$\lambda_m - \overline{\lambda_m} = 2i\beta_m$$

$$y_m(x)\,\overline{y_m(x)} = (f_m(x) + ig_m(x))(f_m(x) - ig_m(x))$$

$$y_m(x)\,\overline{y_m(x)} = f_m^2(x) + g_m^2(x)$$

Substitute in (1)

$$(1) \Rightarrow 2i\beta_m \int_a^b f_m^2(x) + g_m^2(x)dx = 0$$

Since $y_m(x) \neq 0$, we have $f_m^2(x) + g_m^2(x) \neq 0$

$$\Rightarrow \int_a^b f_m^2(x) + g_m^2(x)dx \neq 0$$

Also, $2i \neq 0$

$$\Rightarrow \beta_m = 0$$

$$\lambda_m = \alpha_m + i\beta_m$$

$$\Rightarrow \lambda_m = \alpha_m$$

λ_m is real.

Note:

A Fredholm equation with non-symmetric kernel may possess characteristic numbers which are not real.

<u>Theorem:</u>

Any function $f(x)$ can be generated from a continuous function $\bar{\phi}(x)$ by the operation $\int_a^b k(x,\xi)\ \bar{\phi}(\xi)\ d\xi$ where $k(x,\xi)$ is continuous real and symmetric, so that $f(x) = \int_a^b k(x,\xi)\ \bar{\phi}(\xi)\ d\xi$ for some continuous function $\bar{\phi}$ can be represented over (a,b) by a L. C of the homogenous fredholm equation $y(x) = \lambda \int_a^b k(x,\xi)\, y(\xi)\, d\xi$ with $k(x,\xi)$ as its kernel.

Note:

By the previous theorem $f(x)$ can be written as $f(x) = \sum_n A_n y_n(x)$ where $a \leq x \leq b$

We can find the coefficient $A_1, A_2, \ldots$ in the above expression in the following manner

Now,

$$f(x) = \sum_n A_n y_n(x)$$

i.e $f(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x) + \cdots$

Multiply by $y_n(x)$ we get,

$$f(x) y_n(x) = A_1 y_1(x) y_n(x) + A_2 y_2(x) y_n(x) + \cdots + A_n y_n(x) y_n(x) + \cdots$$

Integrating from a to b we get,

$$\int_a^b f(x) y_n(x)\, dx = A_1 \int_a^b y_1(x) y_n(x)\, dx + A_2 \int_a^b y_2(x) y_n(x) dx + \cdots +$$

$$A_n \int_a^b y_n^2(x) dx \ + \cdots$$

$$\int_a^b f(x) y_n(x)\, dx = 0 + 0 + \cdots + A_n \int_a^b y_n^2(x) dx$$

$$\Rightarrow \int_a^b f(x) y_n(x)\, dx = A_n \int_a^b y_n^2(x) dx$$

$$A_n = \frac{\int_a^b f(x) y_n(x)\, dx}{\int_a^b y_n^2(x) dx}$$

Note:

If there are only finite number of characteristic functions then the function generated by the operator $\int_a^b k(x, \xi)\ \bar{\phi}(\xi)\, d\xi$ form a very restricted class

Example:

Let $k(x, \xi) = \sin(x + \xi)$ and let (a, b) be the interval $(0, 2\pi)$

Soln:

Let $f(x) = \int_0^{2\pi} k(x,\xi)\ \bar{\phi}(\xi)\ d\xi$ ______(1)

Here $k(x,\xi) = \sin(x+\xi)$

$= \sin x \cos\xi + \cos x \sin\xi$

$$k(x,\xi) = f_1(x)g_1(\xi) + f_2(x)g_2(\xi)$$

Where $f_1(x) = \sin x\,, f_2(x) = \cos x\ , g_1(\xi) = \cos\xi\,, g_2(\xi) = \sin\xi$

$$\Rightarrow k(x,\xi) = \sum_{n=1}^{2} f_n(x)g_n(\xi)$$

i.e., $k(x,\xi)$ is separable

$$f(x) = \int_0^{2\pi} (\sin x \cos\xi + \cos x \sin\xi\ )\ \bar{\phi}(\xi)\ d\xi$$

$$= \left[\int_0^{2\pi} [\cos\xi\, \bar{\phi}(\xi)\ d\xi]\right] \sin x + \left[\int_0^{2\pi} [\sin\ \xi\, \bar{\phi}(\xi)\ d\xi]\right] \cos x$$

$$\Rightarrow f(x) = c_1 \sin x + c_2 \cos x$$

Where

$c_1 = \int_0^{2\pi} \cos\xi\, \bar{\phi}(\xi)\ d\xi\ ;\quad c_2 = \int_0^{2\pi} \sin\xi\, \bar{\phi}(\xi)\ d\xi$

Therefore $f(x)$ can generate only functions of the form $f(x) = c_1 \sin x + c_2 \cos x$ regardless of the form of $\bar{\phi}$

The characteristic function of the associated homogenous fredholm integral equation

$y(x) = \lambda \int_0^{2\pi} \sin(x+\xi)\, y(\xi) d\xi$ __(2) can be easily found by the methods in the previous section

Now,

$$y(x) = \lambda \int_0^{2\pi} \sin(x+\xi)\, y(\xi) d\xi$$

$$\Rightarrow y(x) = \lambda \left\{ \left[\int_0^{2\pi} \cos\xi \, y(\xi) \, d\xi \right] \sin x + \left[\int_0^{2\pi} \sin\xi \, y(\xi) \, d\xi \right] \cos x \right\}$$

$$\Rightarrow y(x) = \lambda[c_1 \sin x \ + c_2 \cos x]$$

$$\Rightarrow y(x) = \lambda c_1 \sin x + \lambda c_2 \cos x \ \_\_\_\_\_(3)$$

Multiply (3) by $g_1(x)$ and integrate between $(0,2\pi)$

$$\int_0^{2\pi} y(x) g_1(x) \, dx = \lambda \int_0^{2\pi} c_1 \sin x \ g_1(x) dx + \lambda \int_0^{2\pi} c_2 \cos \ x \ g_1(x) dx$$

$$\int_0^{2\pi} y(x) \cos \ x \, dx = \lambda \int_0^{2\pi} c_1 \sin x \ \cos \ x \, dx + \lambda \int_0^{2\pi} c_2 \cos \ x \ \cos \ x \, dx$$

$$= \lambda c_1 \int_0^{2\pi} \frac{\sin 2x}{2} \, dx + \lambda c_2 \int_0^{2\pi} \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{\lambda c_1}{2} \left[-\frac{\cos 2x}{2} \right]_0^{2\pi} + \frac{\lambda c_2}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{2\pi}$$

$$= \frac{\lambda c_1}{2} \left[-\frac{\cos 4\pi}{2} + \frac{\cos 0}{2} \right] + \frac{\lambda c_2}{2} \left[2\pi + \frac{\sin 4\pi}{2} - 0 - \frac{\sin 0}{2} \right]$$

$$= \frac{\lambda c_1}{2} \left[-\frac{1}{2} + \frac{1}{2} \right] + \frac{\lambda c_2}{2} [2\pi]$$

$$= \pi \lambda c_2$$

$$\int_0^{2\pi} y(x) \cos \ x \, dx = \pi \lambda c_2$$

$$c_1 - \lambda \pi c_2 = 0 \ \_\_\_\_\_\_\_\_(4)$$

Multiply (3) by $g_2(x)$ and integrate between $(0,2\pi)$

$$\int_0^{2\pi} y(x) g_2(x) \, dx = \lambda \int_0^{2\pi} c_1 \sin x \ g_2(x) dx + \lambda \int_0^{2\pi} c_2 \cos \ x \ g_2(x) dx$$

$$\int_0^{2\pi} y(x) \sin \ x \, dx = \lambda \int_0^{2\pi} c_1 \sin x \ \sin x \, dx + \lambda \int_0^{2\pi} c_2 \sin x \ \cos \ x \, dx$$

$$= \lambda c_1 \int_0^{2\pi} \frac{1-\cos 2x}{2}\, dx + \lambda c_2 \int_0^{2\pi} \left(\frac{\sin 2x}{2}\right) dx$$

$$= \frac{\lambda c_1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[-\frac{\cos 2x}{2}\right]_0^{2\pi}$$

$$= \frac{\lambda c_1}{2}\left[2\pi - \frac{\sin 4\pi}{2}\right] - [0 - \sin 0] + \frac{\lambda c_2}{2}\left[-\frac{1}{2} + \frac{1}{2}\right]$$

$$= \pi\lambda c_1$$

$$\int_0^{2\pi} y(x) \sin\, x\, dx = \pi\lambda c_1$$

$$c_2 - \pi\lambda c_1 = 0$$

$$\lambda\pi c_1 - c_2 = 0 \_\_\_\_\_\_\_\_ (5)$$

Solving (4) and (5), the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & -\pi\lambda \\ \pi\lambda & -1 \end{vmatrix} = -1 + \pi^2\lambda^2$$

The unique solution exists iff $-1 + \pi^2\lambda^2 \neq 0$

Iff $\pi^2\lambda^2 \neq 1$

Iff $\lambda^2 \neq \frac{1}{\pi^2}$

Iff $\lambda \neq \pm\frac{1}{\pi}$

The unique solution exists iff $\lambda = \pm\frac{1}{\pi}$

Here the numbers $\lambda = \frac{1}{\pi}$, $\lambda = -\frac{1}{\pi}$ are the characteristic numbers

When $\lambda = \frac{1}{\pi}$ then equation (4) becomes,

$$c_1 - c_2 = 0$$

$$c_1 = c_2$$

Substitute this in equation (3) we get,

$$y(x) = \lambda c_1 \sin x + \lambda c_1 \cos x$$

$$y(x) = \frac{1}{\pi} c_1 [\sin x + \cos x]$$

And this is the characteristics function corresponding to $\lambda = \frac{1}{\pi}$

When $\lambda = \frac{-1}{\pi}$ then equation (4) becomes,

$$c_1 + c_2 = 0$$

$$c_1 = -c_2 \ or \ c_2 = -c_1$$

Substitute this in equation (3) we get,

$$y(x) = \lambda c_1 \sin x - \lambda c_1 \cos x$$

$$y(x) = \frac{-1}{\pi} c_1 [\sin x - \cos x]$$

And this is the characteristics function corresponding to $\lambda = \frac{-1}{\pi}$

Here the characteristic function of $y(x) = \lambda \int_0^{2\pi} \sin(x + \xi)\, y(\xi) d\xi$ are the multiples of the functions $y_1(x) = \sin x + \cos x$, $y_2(x) = \sin x - \cos x$

Now any function of the form $f(x) = c_1 \sin x + c_2 \cos x$ generated by $\int_0^{2\pi} \sin(x + \xi)\, \bar{\phi}(\xi) d\xi$ can be expressed as a L. C. of $y_1(x)$ and $y_2(x)$

$$f(x) = l(\sin x + \cos x) + m(\sin x - \cos x) \_\_\_\_\_\_ (6)$$

But $f(x) = c_1 \sin x + c_2 \cos x \_\_\_\_\_ (7)$

Comparing (6) and (7)

$$c_1 = l + m$$

$$c_2 = l - m$$

$$\Rightarrow c_1 + c_2 = 2l$$

$$l = \frac{c_1 + c_2}{2}$$

$$c_1 - c_2 = 2m$$

$$m = \frac{c_1 - c_2}{2}$$

$$f(x) = \frac{c_1 + c_2}{2}(\sin x + \cos x) + \frac{c_1 - c_2}{2}(\sin x - \cos x)$$

Finding the solution of non-homogenous Fredholm equation of second kind using Hilbert's Schmidt Theory

We can find a continuous solution of the non-homogenous fredholm equation of second kind

$$y(x) = F(x) + \lambda \int_a^b k(x,\xi)y(\xi)d\xi$$

Where $F(x)$ is a given continuous real function.

Let us assume that we have found all members of the set of characteristic function $y_n(x)$ homogenous equation $y(x) = \lambda \int_a^b k(x,\xi)y(\xi)d\xi$ where $k(x,\xi)$ is continuous real and symmetric.

With this knowledge we can find a continuous solution of corresponding non-homogenous fredholm equation

We also assume that

1. The characteristic no. have been ordered with respect to magnitude
2. Characteristic no's corresponding to k independent characteristic functions have been counted k times
3. Subset of independent characteristic function have been orthogonalized
4. The arbitrary multiplicative constants associated with each characteristic function is so chose such that the function is normalised over the interval (a, b)

Thus, we can write $\phi_n = c_n y_n$ where the normalising factor c_n is given by

$$c_n = \frac{1}{\sqrt{\int_a^b \left(y_n(x)\right)^2 dx}}$$

$$\int_a^b \phi_n^2(x)dx = \int_a^b c_n^2\, y_n^2\,(x)dx$$

$$= \int_a^b \frac{1}{\int_a^b \left(y_n(x)\right)^2 dx}\; y_n^2(x)dx$$

$$\int_a^b \phi_n^2(x)dx = 1$$

Hence the expansion of $f(x)$ in the series of the normalized characteristic function takes the simple form $f(x) = \sum_n a_n\phi_n(x)$

To find the co-efficients a_1 and a_2

Now,

$$f(x) = \sum_n a_n\phi_n(x) \_\_\_\_\_\_\_ (a)$$

$$\Rightarrow f(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x) + \cdots$$

Mulitply by $\phi_n(x)$

$$f(x)\phi_n(x) = a_1\phi_1(x)\phi_n(x) + a_2\phi_2(x)\phi_n(x) + \cdots + a_n\phi_n^2(x) + \cdots$$

Integrating,

$$\int_a^b f(x)\phi_n(x)dx = \int_a^b a_n\phi_n^2(x)\, dx$$

$$= a_n \int_a^b \phi_n^2(x)\, dx$$

$$\int_a^b f(x)\phi_n(x)dx = a_n$$

$$a_n = \int_a^b f(x)\phi_n(x)dx$$

By the basic theorem we have

$$f(x) = \sum_n A_n y_n$$

Comparing with (a) we get

$$\sum a_n \phi_n(x) = \sum A_n y_n(x)$$

$$a_n \phi_n(x) = A_n y_n(x)$$

$$A_n = \frac{a_n}{y_n(x)} \phi_n(x)$$

$$A_n = \frac{a_n}{y_n(x)} c_n y_n(x)$$

$$A_n = a_n c_n$$

These relation allows the transition from the expression involving non-normalized characteristic function.

If the function $y(x) = F(x) + \lambda \int_a^b k(x,\xi)y(\xi)d\xi$ possess a continuous solution $y(x)$, then the function $y(x) - F(x)$ is generated by the operation $\int_a^b k(x,\xi)\lambda\, y(\xi)d\xi$ and hence it can be represented by a series of normalized characteristic function $\phi_n(x)$ where $n = 1, 2, \ldots$ of the form $y(x) - F(x) = \sum_n a_n \phi_n(x)\ \ (a \le x \le b)$ and the coefficient of a_n aare given by,

$$a_n = \int_a^b [y(x) - F(x)]\phi_n(x)dx$$

$$a_n = \int_a^b y(x)\phi_n(x)dx - \int_a^b F(x)\, \phi_n(x)dx$$

$$a_n = d_n - f_n$$

Where $d_n = \int_a^b y(x)\phi_n(x)dx$ and $f_n = \int_a^b F(x)\, \phi_n(x)dx$

Now we have to find out the unknowns $d_1, d_2, \ldots, d_n$ where $d_n = \int_a^b y(x)\phi_n(x)dx$

Consider the equation,

$$y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi$$

Multiply by $\phi_n(x)$ and integrated within a to b, we get,

$$\int_a^b y(x)\phi_n(x)dx = \int_a^b F(x)\phi_n(x)\,dx + \lambda \int_a^b \phi_n(x)\left[\int_a^b k(x,\xi)y(\xi)d\xi\right]dx$$

$$d_n = f_n + \lambda \int_a^b \phi_n(x)\left[\int_a^b k(x,\xi)y(\xi)d\xi\right]dx$$

Consider, $\int_a^b \phi_n(x)\left[\int_a^b k(x,\xi)y(\xi)d\xi\right]dx$

$$= \int_a^b y(\xi)\left[\int_a^b k(x,\xi)\phi_n(x)dx\right]d\xi$$

$$= \frac{1}{\lambda_n}\int_a^b y(\xi)\phi_n(\xi)d\xi$$

$$= \frac{d_n}{\lambda_n}$$

Hence $d_n = f_n + \frac{\lambda}{\lambda_n} d_n$

Also we have, $a_n = d_n - f_n$

i.e. $d_n = a_n + f_n$

Here $a_n + f_n = f_n + \frac{\lambda}{\lambda_n}(a_n + f_n)$

$$a_n\left[1 - \frac{\lambda}{\lambda_n}\right] = f_n + \frac{\lambda}{\lambda_n} f_n - f_n$$

$$= \frac{\frac{\lambda}{\lambda_n} f_n}{1 - \frac{\lambda}{\lambda_n}} = \frac{\lambda}{\lambda_n} f_n * \frac{\lambda_n}{\lambda_n - \lambda}$$

$$a_n = \frac{\lambda}{\lambda_n - \lambda} f_n \; if \; \lambda \neq \lambda_n$$

Hence the required solution is

$$y(x) = F(x) + \lambda \sum_{n} \frac{f_n}{\lambda_n - \lambda} \phi_n(x) \_\_\_\_\_\_\_\_\_\_ (1);\ \lambda \neq \lambda_n$$

Where $f_n = \int_a^b F(x)\phi_n(x)dx$

We see that the constants function is the coefficient in the expansion $F(x) = \sum_n f_n \phi_n(x)$, if $F(x)$ is written like this

The expansion (1) exists uniquely iff $\lambda \neq \lambda_n$

i.e., iff λ does not take a characteristic value

Let λ_k be the kth characteristic number. If $\lambda = \lambda_k$, the solution of (1) is not existent unless $f_k = 0$

i.e. Unless $F(x)$ is orthogonal to the corresponding characteristic function

If $\lambda = \lambda_k$ and $f_k = 0$

Consider $d_n = f_n + \frac{\lambda}{\lambda_n} d_n$

Put $n = k$

$$d_k = f_k + \frac{\lambda}{\lambda_k} d_k$$

$$d_k = 0 + \frac{\lambda}{\lambda_k} d_k = d_k$$

Which is a trivial identity when $n = k$ and hence it imposes no restriction to d_k. The coefficient of $\phi_k(x)$ in (1) arbitrary which formally assumes the form $\frac{0}{0}$, so that the equation $y(x) = F(x) + \lambda \int_a^b k(x,\xi)y(\xi)d\xi$ possess an infinitely many solution differing from each other by arbitrary multiple of $\phi_n(x)$

If λ assumes the characteristic value and $F(x)$ is not orthogonal to corresponding characteristic function then no solution exists.

Note:

(i) By the virtue of $a_n\phi_n = A_n y_n$ and $A_n = a_n c_n$ normalization of the characteristic function is unnecessary. In the sense that, equation (1) can be replaced by $y(x) = F(x) + \lambda \sum_n \frac{F_n}{\lambda_n - \lambda} y_n(x)$ $(\lambda \neq \lambda_n)$ where $F_n = \int_a^b F(x) y_n(x) dx$ where $n = 1, 2, 3, \dots$

(ii) Consider the Fredholm equation of first kind given by $F(x) = \int_a^b k(x,\xi) y(\xi) d\xi$ ____(1) with a continuous, real and symmetric kernel where F is a prescribed continuous function and y is to be determined.

It follows from the basic expansion that (1) has no continuous solution unless $F(x)$ can be expressed as a L. C. of the characteristic function corresponding to the associated homogenous equation of second kind $y(x) = \lambda \int_a^b k(x,\xi) y(\xi) d\xi$

Example:

Solve the equation $F(x) = \int_0^{2\pi} y(\xi) \sin(x+\xi) d\xi$

Soln:

The given equation is $F(x) = \int_0^{2\pi} y(\xi) \sin(x+\xi) d\xi$

Here $k(x,\xi) = \sin(x+\xi)$ and it is fredholm equation of the first kind

Now,

$$F(x) = \int_0^{2\pi} y(\xi)\, [\sin x \cos\xi + \cos x \sin\xi] d\xi$$

$$= \int_0^{2\pi} y(\xi) \sin x \cos\xi d\xi + \int_0^{2\pi} y(\xi) \cos x \sin\xi d\xi$$

$$= \sin x \int_0^{2\pi} y(\xi) \cos\xi d\xi + \cos x \int_0^{2\pi} y(\xi) \sin\xi d\xi \_\_\_\_\_(1)$$

This relation can be satisfied only if $F(x)$ is prescribed as the L. C. of $\sin x$ and $\cos x$ or as a L.C. of the characteristic function y_1 and y_2

$$y(x) = \lambda \int_0^{2\pi} y(\xi) \sin(x + \xi) d\xi$$

With $y_1 = \sin x + \cos x$, $y_2 = \sin x - \cos x$ corresponding to the characteristic number $\lambda_1 = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$

If $F(x)$ is prescribed as $F(x) = A \sin x + B \cos x$ then the equation (1) is satisfied by the any function y for which

$$A = \int_0^{2\pi} y(\xi) \cos \xi d\xi \_\_\_\_\_\_\_\_\_(2)$$

And

$$B = \int_0^{2\pi} y(\xi) \sin \xi d\xi \_\_\_\_\_\_\_\_(3)$$

We can find one such function as $y(x) = \frac{1}{\pi}(A \cos x + B \sin x)$______(4)

If we add to (4) any function which is orthogonal to both $\sin x$ and $\cos x$ and hence to the characteristic function y_1 and y_2 over 0 to 2π. Then the condition (2) and (3) will still be satisfied so that the solution is not unique

No solution exists unless $F(x)$ is prescribed in the form $F(x) = A \sin x + B \cos x$

Let us consider the case that $F(x) = \int_a^b k(x, \xi) y(\xi) d\xi$ possess a continuous solution. Then $F(x)$ is generated from $y(x)$ by the operation $\int_a^b k(x, \xi) y(\xi) d\xi$ and hence it can be expanded in a series $F(x) = \sum_n f_n \phi_n(x)$, $a \leq x \leq b$ where $f_n = \int_a^b F(x) \phi_n(x) dx$ where ϕ_n is the nth characteristic function of $y(x) = \lambda \int_a^b k(x, \xi) y(\xi) d\xi$.

This series $F(x)$ may be finite or infinite. Since ϕ_n satisfied the equation, $\phi_n(x) = \lambda_n \int_a^b k(x, \xi) \phi_n(\xi) d\xi$ and $F(x) = \int_a^b k(x, \xi) y(\xi) d\xi$ and $F(x) = \sum_n f_n \phi_n(x)$ we have

$$\int_a^b k(x, \xi) y(\xi) d\xi = \sum_n f_n \lambda_n \int_a^b k(x, \xi) \phi_n(\xi) d\xi$$

$$\int_a^b k(x,\xi)y(\xi)d\xi - \sum_n f_n\lambda_n \int_a^b k(x,\xi)\phi_n(\xi)d\xi = 0$$

i.e.

$$\int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi = 0$$

This condition is satisfied iff $y(x)$ is of the form $y(x) = \sum_n \lambda_n f_n \phi_n(x) + \bar{\phi}(x)$ where $\bar{\phi}$ is the solution of the equation $\int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi = 0$

Hence if $F(x) = \int_a^b k(x,\xi)y(\xi)d\xi$ possess a continuous solution then the solution must be of the form $y(x) = \sum_n \lambda_n f_n \phi_n(x) + \bar{\phi}(x)$ where ϕ is any continuous function satisfying $\int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi = 0$.

From the homogeneity of the equation $\int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi = 0$. It is clear that either this equation is satisfied by the trivial function $\phi(x) = 0$ or it possess infinitely many solution,

Consider the equation $\int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi = 0$

Multiply both the sides by $\phi_n(x)$ and integrate within a to b we get,

$$\int_a^b \phi_n(x) \int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi \; dx = 0$$

$$\int_a^b \bar{\phi}(\xi) \left[\int_a^b k(x,\xi)\phi_n(x)dx\right] d\xi = 0$$

$$\Rightarrow \int_a^b \bar{\phi}(\xi) \frac{\phi_n(\xi)}{\lambda_n} d\xi = 0$$

$$\Rightarrow \frac{1}{\lambda_n}\int_a^b \bar{\phi}(\xi)\, \phi_n(\xi)d\xi = 0$$

$$\Rightarrow \int_a^b \bar{\phi}(\xi)\, \phi_n(\xi)d\xi = 0$$

Hence if $\int_a^b k(x,\xi)\bar{\phi}(\xi)d\xi\ \ dx = 0$ possess a non trivial solution then that solution must be orthogonal to all characteristics function $\phi(x)$

Note:

(i) If this set of functions is finite, then infinitely many linearly independent functions satisfying this condition exists.

(ii) If the functions ϕ_n's comprise an infinite complete set over (a, b) then no continuous non trivial function can be simultaneously orthogonal to all functions of the sers, so that in this case the function $\bar{\phi}$ in $y(x) = \sum_n \lambda_n f_n \phi_n(x) + \bar{\phi}\,(x)$ must be identically zero.

(iii) If the non-homogenous equation of the second kind $y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi)\, d\xi$ possess a continuous solution is unique unless λ assumes a characteristic value and is given by, $y(x) = F(x) + \lambda \sum_n \frac{f_n}{\lambda_n - \lambda} y_n(x)\ (\lambda_n \neq \lambda)$

(iv) If the equation $y(x) = \int_a^b k(x,\xi) y(\xi)\, d\xi$ of the first kind possess a continuous solution then it is given by $y(x) = \sum_n \lambda_n f_n \phi_n(x) + \bar{\phi}\,(x)$ it is (or not) uniquely defined according as $\int_a^b k(x,\xi)\bar{\phi}\,(\xi) d\xi$ does not possess a non-trivial solution.

Section 3.9

Iterative methods for solving the equations of second kind

We can solve the integral equations of second kind by the method of successive approximation.

Consider the Fredholm of second kind given by $y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi$_____(1) where f and k are continuous.

Let $y^{(0)}$ be the initial approximation. Replace y under the integral sign by the initial approximation $y^{(0)}$ then the next approximation given by

$$y^{(1)}(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi)\, d\xi \_\_\_\_\_\_\_(2)$$

Similarly, the next approximation $y^{(2)}(x)$ is given by,

$$y^{(2)}(x) = F(x) + \lambda \int_a^b k(x,\xi) y^{(1)}(\xi)\, d\xi$$

Continuing this process we get,

$$y^{(n)}(x) = F(x) + \lambda \int_a^b k(x,\xi) y^{(n-1)}(\xi)\, d\xi \_\_\_\_\_\_\_\_ (3)$$

We can use the same method for Volterra equation also where the upper limit b is replaced by the current variable x.

Our aim is to determine under what condition the successive approximation gives the continuous solution.

Let us replace y by $y^{(1)}$ in the equation (1) as follows

First replace the current variable x by the dummy variable ξ and replace ξ by another dummy variable ξ_1 then we get,

$$(2) \Rightarrow y^{(1)}(\xi) = F(\xi) + \int_a^b k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1$$

The next approximation is

$$y^{(2)}(x) = F(x) + \lambda \int_a^b k(x,\xi) y^{(1)}(\xi)\, d\xi$$

$$y^{(2)}(x) = F(x) + \lambda \int_a^b k(x,\xi) \left[F(\xi) + \int_a^b k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1 \right] d\xi$$

$$= F(x) + \lambda \int_a^b k(x,\xi) F(\xi) d\xi + \lambda^2 \int_a^b k(x,\xi) \int_a^b k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1\, d\xi$$

To find $y^{(2)}(\xi)$, replace x by ξ, ξ by ξ_1 and ξ_1 by ξ_2

$$y^{(2)}(\xi) = F(\xi) + \lambda \int_a^b k(\xi,\xi_1)\; F(\xi_1) d\xi_1 + \lambda^2 \int_a^b k(\xi,\xi_1) \int_a^b k(\xi_1,\xi_2) y^{(0)}(\xi_2) d\xi_2\, d\xi_1$$

The next approximation is

$$y^{(3)}(x) = F(x) + \lambda \int_a^b k(x,\xi) y^{(2)}(\xi)\, d\xi$$

$$= F(x) + \lambda \int_a^b k(x,\xi) \left[F(\xi) + \int_a^b k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1 + \lambda^2 \int_a^b k(\xi,\xi_1) \int_a^b k(\xi_1,\xi_2) y^{(0)}(\xi_2) d\xi_2\, d\xi_1 \right] d\xi$$

$$= F(x) + \lambda \int_a^b k(x,\xi) F(\xi) d\xi + \lambda^2 \int_a^b k(x,\xi) \int_a^b k(\xi,\xi_1) F(\xi_1) d\xi_1 d\xi + \lambda^3 \int_a^b k(x,\xi) \int_a^b k(\xi,\xi_1) \int_a^b k(\xi_1,\xi_2) y^{(0)}(\xi_2) d\xi_2\, d\xi_1 d\xi \quad \text{\_\_\_\_(4)}$$

Define an integral operator κ by

$$\kappa f(x) = \int_a^b k(x,\xi) F(\xi) d\xi \quad \text{\_\_\_\_(5)}$$

By using this operator equation (1) takes the symbolic form,

$$y(x) = F(x) + \lambda \kappa y(x) \text{\_\_\_\_\_\_\_\_\_\_(6)}$$

Equation (3) takes the form

$$y^{(n)}(x) = F(x) + \lambda \kappa y^{(n-1)}(x)$$

Similarly,

$$y^{(1)}(x) = F(x) + \lambda \kappa y^{(0)}(x)$$

$$y^{(2)}(x) = F(x) + \lambda \kappa (F(x)) + \lambda^2 \kappa^2 y^{(0)}(x)$$

$$y^{(3)}(x) = F(x) + \lambda \kappa F(x) + \lambda^2 \kappa^2 F(x) + \lambda^3 \kappa^3 y^{(0)}(x)$$

$$\vdots$$

Proceeding like this we get,

$$y^{(n)}(x) = F(x) + \lambda \kappa F(x) + \lambda^2 \kappa^2 F(x) + \lambda^3 \kappa^3 F(x) + \lambda^4 \kappa^4 F(x) + \cdots + \lambda^{n-1} \kappa^{n-1} F(x) + \lambda^n \kappa^n y^{(0)}(x)$$

Here $\lambda^n \kappa^n y^{(0)}(x) = R_n(x)$

The solution of the given equation can be expressed as

$$y(x) = F(x) + \sum_{n=1}^{\alpha} \lambda^n \kappa^n F(x) \_\_\_\_\_\_(7) \;\; as\; n \to \infty$$

The only thing to be determined is the condition under which the expression $R_n(x) \to 0$ and under which condition the series (7) convergent and gives a continuous solution.

We have $k(x,\xi)$ is continuous for all $x \in (a,b)$

$\Rightarrow k\ (x,\xi)$ is bounded in (a,b)

There exists a positive constant M such that $k(x,\xi) \leq M$ _______(8)

Similarly, $F(x)$ is continuous in (a,b)

$\Rightarrow$ It is bounded and hence there exists a positive constant m such that $|\ F(x)\ | \leq m$ _____(9)

Let us assume that the magnitude of the initial approximation $y^{(0)}(x)$ is bounded in (a,b)

Hence there exists a positive constant c such that $|\ y^{(0)}(x)\ | \leq c$ in (a,b)_______(10)

Now,

$$\left|\kappa y^{(0)}(x)\right| = \left|\int_a^b k(x,\xi) y^{(0)}(\xi)\, d\xi\right|$$

$$\leq \int_a^b \left|k(x,\xi) y^{(0)}(\xi)\right| d\xi$$

$$= \int_a^b |k(x,\xi)|\left|y^{(0)}(\xi)\right| d\xi$$

$$\leq \int_a^b Mc\; d\xi$$

$$= Mc \int_a^b d\xi$$

$$= Mc[\xi]_a^b$$

$$= Mc(b-a)$$

$$\left|\kappa y^{(0)}(x)\right| \leq Mc(b-a)$$

Similarly, we can find that,

$$\left|\kappa^2\, y^{(0)}(x)\right| = \left|\int_a^b k(x,\xi)\int_a^b k(\xi,\xi_1)y^{(0)}(\xi_1)d\xi_1\, d\xi\right|$$

$$\left|\kappa^2\, y^{(0)}(x)\right| \leq M^2c(b-a)^2$$

More generally by iteration we have

$$\left|\kappa^n\, y^{(0)}(x)\right| \leq M^nc(b-a)^n$$

Similarly,

$$|\kappa^n F(x)| \leq M^n m(b-a)^n$$

We have

$$R_n(x) = \lambda^n\kappa^n y^{(0)}(x)$$

$$\Rightarrow |R_n(x)| = \left|\lambda^n\kappa^n y^{(0)}(x)\right|$$

$$= |\lambda^n|\left|\kappa^n y^{(0)}(x)\right|$$

$$\leq |\lambda^n|M^nc(b-a)^n$$

For large values of n, $R_n(x) \to 0$ if $|\lambda| < \frac{1}{M(b-a)}$

Also we have by equation (7)

$$y(x) = F(x) + \sum_{n=1}^{\infty}\lambda^n\kappa^n F(x)$$

Now,

$$\left|F(x) + \sum_{n=1}^{\infty}\lambda^n\kappa^n F(x)\right| \leq |F(x)| + \left|\sum_{n=1}^{\infty}\lambda^n\kappa^n F(x)\right|$$

$$\leq |F(x)| + \sum_{n=1}^{\infty} |\lambda^n \kappa^n F(x)|$$

$$= |F(x)| + \sum_{n=1}^{\infty} |\lambda^n||\kappa^n F(x)|$$

$$\leq m + \sum_{n=1}^{\infty} |\lambda^n| M^n m (b-a)^n$$

$$= m\left[1 + \sum_{n=1}^{\infty} |\lambda^n| M^n (b-a)^n\right]$$

This is a geometric series and this geometric series converges if $|\lambda| M(b-a) < 1$

i.e. $|\lambda| < \frac{1}{M(b-a)}$

$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \kappa^n F(x)$ converges absolutely and hence uniformly in (a, b) if $|\lambda| < \frac{1}{M(b-a)}$

We can show that the series $y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \kappa^n F(x)$ satisfies the integral equation $y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi$ and hence it represents the continuous solution of (1) when $|\lambda| < \frac{1}{M(b-a)}$ and k is continuous.

Note:

Let λ_1 be a characteristic number of (1). Then equation (7) fails to converge if $|\lambda|$ is equal to the absolute value of the smallest characteristic number λ_1

i.e. The series in (7) converges if $|\lambda| < \lambda_1$

Hence we have $|\lambda_1| \geq \frac{1}{M(b-a)}$

Hence $\frac{1}{M(b-a)}$ is the lower bound of the magnitude of the smallest characteristic number λ_1

When k is real and symmetric we can show that $\lambda_1 \geq \frac{1}{\sqrt{\int_a^b \int_a^b k(x,\xi)^2 dx d\xi}}$

Another method to solve the Fredholm equation of second kind

Consider the Fredholm equation of second kind given by,

$$y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi$$

$$y(x) = F(x) + \lambda \kappa y(x)$$

Let $\mathcal{I}$ be the identity operator

$$\Rightarrow \mathcal{I}\big(y(x)\big) = F(x) + \lambda \kappa y(x)$$

$$\Rightarrow \mathcal{I}\big(y(x)\big) - \lambda \kappa y(x) = F(x)$$

$$\Rightarrow (\mathcal{I} - \lambda \kappa) y(x) = F(x)$$

$$\Rightarrow y(x) = (\mathcal{I} - \lambda \kappa)^{-1} F(x)$$

$$\Rightarrow y(x) = [\mathcal{I} + \lambda \kappa + (\lambda \kappa)^2 + (\lambda \kappa)^3 + \cdots] F(x)$$

$$\Rightarrow y(x) = F(x) + \lambda \kappa F(x) + (\lambda \kappa)^2 F(x) + (\lambda \kappa)^3 F(x) + \cdots$$

$$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \kappa^n F(x)$$

And this equation is valid only if $|\lambda| < |\lambda_1|$

Solving the Volterra equation using Iterative method:

Consider the volterra equation of second kind given by

$$y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi \_\_\_\_\_\_\_\_\_\_ (1)$$

Where f and k are continuous.

Let $y^{(0)}$ be the initial approximation. Then the next approximation is of the form

$$y^{(1)}(x) = F(x) + \lambda \int_a^x k(x,\xi) y^{(0)}(\xi) d\xi \_\_\_\_\_\_\_\_\_\_ (2)$$

Similarly,

$$y^{(2)}(x) = F(x) + \lambda \int_a^x k(x,\xi) y^{(1)}(\xi) d\xi$$

$$\vdots$$

$$y^{(n)}(x) = F(x) + \lambda \int_a^x k(x,\xi) y^{(n-1)}(\xi) d\xi \_\_\_\_\_\_\_\_\_\_(3)$$

Our aim is to find under what condition the successive approximation gives the continuous solution.

Let us replace y by $y^{(1)}$ in equation (1) as follows. Replace current variable x by the dummy variable ξ and ξ by ξ_1. Then we get,

$$(4) \Rightarrow y^{(1)}(\xi) = F(\xi) + \lambda \int_a^x k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1$$

The next approximation is,

$$y^{(2)}(x) = F(x) + \lambda \int_a^x k(x,\xi) y^{(1)}(\xi) d\xi$$

$$= F(x) + \lambda \int_a^x k(x,\xi) \left[F(\xi) + \lambda \int_a^x k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1 \right] d\xi$$

$$= F(x) + \lambda \int_a^x k(x,\xi) F(\xi) d\xi + \lambda^2 \int_a^x k(x,\xi) \int_a^x k(\xi,\xi_1) y^{(0)}(\xi_1) d\xi_1 \, d\xi$$

To find $y^{(2)}(\xi)$, replace x by ξ, ξ by ξ_1 and $\xi_1$by ξ_2

$$y^{(2)}(\xi) = F(\xi) + \lambda \int_a^x k(\xi,\xi_1) F(\xi_1) d\xi_1 + \lambda^2 \int_a^x k(\xi,\xi_1) \int_a^x k(\xi_1,\xi_2) y^{(0)}(\xi_2) d\xi_2 \, d\xi_1$$

The next approximation is

$$y^{(3)}(x) = F(x) + \lambda \int_a^x k(x,\xi) y^{(2)}(\xi) d\xi$$

$$= F(x) + \lambda \int_a^x k(x,\xi) \left[F(\xi) + \lambda \int_a^x k(\xi,\xi_1) F(\xi_1) d\xi_1 + \lambda^2 \int_a^x k(\xi,\xi_1) \int_a^x k(\xi_1,\xi_2) y^{(0)}(\xi_2) d\xi_2 \, d\xi_1 \right] d\xi$$

$$= F(x) + \lambda \int_a^x k(x,\xi) d\xi + \lambda^2 \int_a^x k(x,\xi) \int_a^x k(\xi,\xi_1) F(\xi_1) d\xi_1 \, d\xi + \lambda^3 \int_a^x k(x,\xi) \int_a^x k(\xi,\xi_1) \int_a^x k(\xi_1,\xi_2) y^{(0)}(\xi_2) d\xi_2 \, d\xi_1 d\xi \_\_\_\_(4)$$

Define an integral operator κ_x by

$$\kappa_x f(x) = \int_a^b k(x,\xi) f(\xi) d\xi \_\_\_\_(5)$$

By using this operator equation (1) takes the form

$$y(x) = F(x) + \lambda \kappa_x y(x) \_\_\_\_(6)$$

Equation (2) takes the form

$$y^{(2)}(x) = F(x) + \lambda \kappa_x y^{(1)}(x)$$

Equation (3) takes the form

$$y^{(n)}(x) = F(x) + \lambda \kappa_x y^{(n-1)}(x)$$

Similarly,

$$y^{(1)}(x) = F(x) + \lambda \kappa_x y^{(0)}(x)$$

$$y^{(2)}(x) = F(x) + \lambda \kappa_x F(x) + \lambda^2 \kappa_x^2 y^{(0)}(x)$$

$$y^{(3)}(x) = F(x) + \lambda \kappa_x F(x) + \lambda^2 \kappa_x^2 F(x) + \lambda^3 \kappa_3 y^{(0)}(x)$$

$$\vdots$$

Proceeding like this we get,

$$y^{(n)}(x) = F(x) + \lambda \kappa_x F(x) + \lambda^2 \kappa_x^2 F(x) + \lambda^3 \kappa_x^3 F(x) + \lambda^4 \kappa_x^4 F(x) + \cdots + \lambda^{n-1} \kappa_x^{n-1} F(x) + \lambda^n \kappa_x^n y^{(0)}(x)$$

Let $R_n(x) = \lambda^n \kappa_x^n y^{(0)}(x)$

The solution of the given equation cane be expressed as

$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \kappa_x^n F(x)$ ____(7) as $n \to \infty$

The only thing to be determined is the condition under which the expression $R_n(x) \to 0$ and under which condition the series (7) converges and gives a continuous solution.

We have $k(x, \xi)$ is continuous for all $x \in (a, b)$

$\Rightarrow k(x, \xi)$ is bounded in (a, x)

There exists a positive constant M such that $k(x, \xi) \leq M$_______(8)

Similarly, $F(x)$ is continuous in (a, x)

$\Rightarrow$ It is bounded and hence there exists a positive constant m such that $| F(x) | \leq m$ _____(9)

Let us assume that the magnitude of the initial approximation $y^{(0)}(x)$ is bounded in (a, x).

Hence there exists a positive constant c such that $| y^{(0)}(x) | \leq c$___(10) in (a, x)

Now,

$$|\kappa_x y^{(0)}(x)| = \left| \int_a^x k(x, \xi) y^{(0)}(\xi) d\xi \right|$$

$$\leq \int_a^x |k(x, \xi) y^{(0)}(\xi)| d\xi$$

$$= \int_a^x |k(x, \xi)| |y^{(0)}(\xi)| d\xi$$

$$\leq \int_a^x Mc d\xi$$

$$= Mc \int_a^x d\xi$$

$$= Mc[\xi]_a^x$$

$$= Mc(x - a)$$

$$\left|\kappa_x y^{(0)}(x)\right| \leq Mc(x-a)$$

$$\left|\kappa_x^2 y^{(0)}(x)\right| = \left|\int_a^x k(x,\xi)\int_a^x k(\xi,\xi_1)\, y^{(0)}(\xi_1)d\xi_1 d\xi\right|$$

$$= \left|\int_a^x k(x,\xi)\kappa_x y^{(0)}(\xi)d\xi\right|$$

$$\leq \int_a^x |k(x,\xi)|\left|\kappa_x y^{(0)}(\xi)\right| d\xi$$

$$\leq \int_a^x MMc(\xi-a)d\xi$$

$$= M^2 c\int_a^x (\xi-a)d\xi$$

$$= M^2 c\left[\frac{(\xi-a)^2}{2}\right]_a^x$$

$$= M^2 c\left[\frac{(x-a)^2}{2}\right]$$

$$\left|\kappa_x^2 y^{(0)}(x)\right| \leq M^2 c\left[\frac{(x-a)^2}{2!}\right]$$

$$\left|\kappa_x^n y^{(0)}(x)\right| \leq M^n c\left[\frac{(x-a)^n}{n!}\right]$$

$$R_n(x) = \lambda^n \kappa_x^n y^{(0)}(x)$$

$$|R_n(x)| = \left|\lambda^n \kappa_x^n y^{(0)}(x)\right|$$

$$= |\lambda^n|\left|\kappa_x^n y^{(0)}(x)\right|$$

$$\leq |\lambda^n| M^n c\frac{(x-a)^n}{n!}$$

$$|R_n(x)| \leq |\lambda^n| M^n c\frac{(b-a)^n}{n!}, a \leq x \leq b$$

Similarly,

$$|\lambda^n \kappa_x^n F(x)| \leq |\lambda^n| M^n c \frac{(b-a)^n}{n!} \; for \; a \leq x \leq b$$

The series in equation (7) converges for any finite value of λ

The series (7) converges to a unique continuous solution for the volterra equation for all values of λ in (a, b) in which $F(x)$ and $k(x, \xi)$ are continuous.

Note

The final solution in each case is independent of $y^{(0)}(x)$

Example

Solve the Fredholm equation $y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi$ using method of iteration.

Define an operator

$$\kappa\big(f(x)\big) = \int_0^1 k(x, \xi)\, f(\xi) d\xi$$

$$= \int_0^1 (1 - 3x\xi)\, f(\xi) d\xi$$

The solution of the fredholm equation using iterative method is given by

$$y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \kappa^n \big(F(x)\big)$$

$$\kappa F(x) = \int_0^1 (1 - 3x\xi)\, F(\xi) d\xi$$

$$= \int_0^1 (1 - 3x\xi)\, d\xi$$

$$= \left[\xi - 3x \frac{\xi^2}{2}\right]_0^1$$

$$= 1 - \frac{3x}{2}$$

$$\kappa^2F(x) = \int_0^1 (1-3x\xi)\kappa F(\xi)\, d\xi$$

$$= \int_0^1 (1-3x\xi)(1-\frac{3\xi}{2})\, d\xi$$

$$= \int_0^1 \left(1-\frac{3\xi}{2}-3x\xi+\frac{9x\xi^2}{2}\right) d\xi$$

$$= \left[\xi-\frac{3}{2}\frac{\xi^2}{2}-3x\frac{\xi^2}{2}+\frac{9x}{2}\frac{\xi^3}{3}\right]_0^1$$

$$= 1-\frac{3}{2}x-\frac{3}{4}+\frac{3}{2}x$$

$$\kappa^2F(x) = \frac{1}{4}$$

$$\kappa^3F(x) = \int_0^1 (1-3x\xi)\kappa^2F(\xi)\, d\xi$$

$$= \int_0^1 (1-3x\xi)(\frac{1}{4})\, d\xi$$

$$= \frac{1}{4}\left[\xi-\frac{3x\xi^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left[1-\frac{3x}{2}\right]$$

$$\kappa^4F(x) = \int_0^1 (1-3x\xi)\kappa^3F(\xi)\, d\xi$$

$$= \int_0^1 (1-3x\xi)\frac{1}{4}\left[1-\frac{3x}{2}\right] d\xi$$

$$= \frac{1}{4}\int_0^1 (1-3x\xi)\left[1-\frac{3x}{2}\right] d\xi$$

$$= \frac{1}{4}\int_0^1 \left(1-\frac{3\xi}{2}-3x\xi+\frac{9x\xi^2}{2}\right) d\xi$$

$$= \frac{1}{4}\left(\frac{1}{4}\right)$$

$$\kappa^4 F(x) = \frac{1}{16}$$

The solution of the Fredholm equation becomes

$$y(x) = F(x) + \lambda\kappa F(x) + \lambda^2\kappa^2 F(x) + \lambda^3\kappa^3 F(x) + \lambda^4\kappa^4 F(x) + \cdots$$

$$= 1 + \lambda\left[1 - \frac{3x}{2}\right] + \lambda^2\left(\frac{1}{4}\right) + \lambda^3\left(\frac{1}{4}\left[1 - \frac{3x}{2}\right]\right) + \lambda^4\left(\frac{1}{16}\right) + \cdots$$

$$= \left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right) + \left(1 - \frac{3}{2}x\right)\left(\lambda + \frac{\lambda^3}{4} + \frac{\lambda^5}{16} + \cdots\right)$$

$$= \left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right) + \lambda\left(1 - \frac{3}{2}x\right)\left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right)$$

$$y(x) = \left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right)\left(1 + \lambda\left(1 - \frac{3}{2}x\right)\right)$$

The series $1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots$ is a geometric series with common ration $\frac{\lambda^2}{4}$ and this series converges if

$$\frac{\lambda^2}{4} < 1$$

$$\Rightarrow \lambda^2 < 4$$

$$\Rightarrow \lambda < \pm 2$$

$$\Rightarrow \mid \lambda \mid < 2$$

Now,

$$1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots = \frac{1}{1 - r}$$

$$1 + \frac{\lambda^2}{4} + \left(\frac{\lambda^2}{4}\right)^2 + \cdots = \frac{1}{1 - \frac{\lambda^2}{4}}$$

$$1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2+\cdots=\frac{4}{4-\lambda^2}$$

$$y(x)=\frac{4}{4-\lambda^2}\left[1+\lambda\left(1-\frac{3x}{2}\right)\right]$$

$$=\frac{4}{4-\lambda^2}\left[1+\lambda\left(\frac{2-3x}{2}\right)\right]$$

$$=\frac{4}{4-\lambda^2}\left[\frac{2+2\lambda-3x\lambda}{2}\right]$$

$$y(x)=\frac{4+2\lambda(2-3x)}{4-\lambda^2}$$

This is the solution of the given fredholm equation and this is valid only when $|\lambda|<2$

Another Method

Consider the equation $y(x)=1+\lambda\int_0^1(1-3x\xi)y(\xi)d\xi$

By the method of separable kernel we have a unique solution iff $\lambda\neq\pm2$ and

$$(1-\lambda)c_1+\frac{3}{2}\lambda c_2=\int_0^1 F(x)dx=\int_0^1 dx=[x]_0^1=1$$

$$-\frac{1}{2}\lambda c_1+(1+\lambda)c_2=\int_0^1 xF(x)dx=\int_0^1 x\,dx=\left[\frac{x^2}{2}\right]_0^1=\frac{1}{2}$$

Now,

$$\Delta=\begin{vmatrix}1-\lambda & \frac{3}{2}\lambda\\ -\frac{1}{2}\lambda & 1+\lambda\end{vmatrix}$$

$$=(1-\lambda)(1+\lambda)-\left(-\frac{1}{2\lambda}\right)\left(\frac{3}{2\lambda}\right)$$

$$=1-\frac{\lambda^2}{4}$$

$$\Delta = \frac{4-\lambda^2}{4}$$

$$\Delta c_1 = \begin{vmatrix} 1 & \frac{3}{2}\lambda \\ \frac{1}{2} & 1+\lambda \end{vmatrix} = 1+\lambda-\frac{3}{4}\lambda = 1+\frac{\lambda}{4}$$

$$\Delta c_2 = \begin{vmatrix} 1-\lambda & 1 \\ -\frac{1}{2}\lambda & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}-\frac{\lambda}{2}+\frac{\lambda}{2} = \frac{1}{2}$$

$$c_1 = \frac{\Delta c_1}{\Delta} = \frac{1+\frac{\lambda}{4}}{\frac{4-\lambda^2}{4}} = \frac{4+\lambda}{4-\lambda^2}$$

$$c_2 = \frac{\Delta c_2}{\Delta} = \frac{\frac{1}{2}}{\frac{4-\lambda^2}{4}} = \frac{2}{4-\lambda^2}$$

The solution of the fredholm equation is

$$y(x) = \lambda[c_1 - 3xc_2] + F(x)$$

$$= \lambda\left[\frac{4+\lambda}{4-\lambda^2} - 3x\left(\frac{2}{4-\lambda^2}\right)\right] + 1$$

$$= \frac{4\lambda+\lambda^2-6x\lambda}{4-\lambda^2} + 1$$

$$= \frac{2\lambda(2-3x)+4}{4-\lambda^2}$$

This is the solution of the given Fredholm equation

Fredholm Theory

Consider the Fredholm equation of second kind $y(x) = F(x) + \lambda \int_a^b k(x,\xi)y(\xi)d\xi$

By iterative method we have

$$\kappa f(x) = \int_a^b k(x,\xi) f(\xi) d\xi \_\_\_\_\_\_\_ (1)$$

Replace ξ by ξ_1,

$$\kappa f(x) = \int_a^b k(x,\xi_1) f(\xi_1) d\xi_1 \_\_\_\_\_\_\_ (2)$$

$$\kappa^2 f(x) = \int_a^b k(x,\xi_1) \kappa f(\xi_1) d\xi_1$$

$$= \int_a^b k(x,\xi_1) \left[\int_a^b k(\xi_1,\xi) f(\xi) d\xi \right] d\xi_1$$

$$= \int_a^b \left[\int_a^b k(x,\xi_1)\, k(\xi_1,\xi) d\xi_1 \right] f(\xi) d\xi \_\_\_\_\_\_\_ (3)$$

Define $k_2(x,\xi) = \int_a^b k(x,\xi_1) k(\xi_1,\xi) d\xi_1$

This $k_2(x,\xi)$ is called as iterated kernel

$$(3) \Rightarrow \kappa^2 f(x) = \int_a^b k_2(x,\xi) f(\xi) d\xi \_\_\_\_\_\_\_ (4)$$

From (2)

Now,

$$\kappa^3 f(x) = \int_a^b k(x,\xi_1) \kappa^2 f(\xi_1) d\xi_1$$

$$= \int_a^b k(x,\xi_1) \left[\int_a^b k_2(\xi_1,\xi) f(\xi) d\xi \right] d\xi_1$$

$$= \int_a^b \left[\int_a^b k(x,\xi_1)\, k_2(\xi_1,\xi)\, d\xi_1 \right] f(\xi) d\xi \_\_\_\_\_\_\_ (5)$$

Define $k_3(x,\xi) = \int_a^b k(x,\xi_1) k_2(\xi_1,\xi) d\xi_1$

$$(5) \Rightarrow \kappa^3 f(x) = \int_a^b k_3(x,\xi) f(\xi) d\xi \_\_\_\_\_\_\_ (6)$$

Repeating this process

$$\kappa^n f(x) = \int_a^b k_n(x,\xi) f(\xi) d\xi \text{ \_\_\_\_\_\_\_\_(7)}$$

Where

$$k_{x,\xi} = \int_a^b k(x,\xi_1) k_{n-1}(\xi_1,\xi)\, d\xi_1$$

$k_n(x,\xi)$ is called as n^{th} iterated kernel for $n = 2,3,4,\ldots$ and where we write

$$k_1(x,\xi) \equiv k(x,\xi)$$

It is not difficult to establish the consequent validity of the relation

$$k_{p+q}(x,\xi) = \int_a^b k_p(x,\xi_1) k_q(\xi_1,\xi) d\xi_1$$

For any positive integer p and q

Further if $k(x,\xi)$ is bounded in (a,b) then $|k(x,\xi)| \leq M$ in (a,b) then it follows easily that $|k_n(x,\xi)| \leq M^n(b-a)^{n-1}$ for the values of x and ξ in (a,b)

With the notation of equation (7) the series $y(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \kappa^n F(x)$ which is the solution of the equation $y(x)$ for sufficiently small values of $|\lambda|$ takes the form

$$y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi \text{\_\_\_\_\_\_\_\_\_(8)}$$

$$= F(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b k_n(x,\xi) F(\xi) d\xi$$

$$= F(x) + \lambda \int_a^b \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,\xi) F(\xi) d\xi \text{\_\_\_\_\_\_\_(9)}$$

Define

$$\Gamma(x,\xi:\lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,\xi)$$

$$= k(x,\xi) + \lambda k_2(x,\xi) + \lambda^3 k_3(x,\xi) + \cdots \text{\_\_\_\_\_\_\_\_\_\_}(10)$$

Substitute (10) in (9) we get,

$$y(x) = F(x) + \lambda \int_a^b \Gamma(x,\xi:\lambda)\; F(\xi) d\xi \text{\_\_\_\_\_\_\_\_\_}(11)$$

This $\Gamma(x,\xi:\lambda)$ is called as the reciprocal kernel or resolvent kernel associated with the kernel $k(x,\xi)$ in the interval (a,b)

The series $\sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,\xi)$ is called the Neumann series,

Note 1:

The series $\sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,\xi)$ converges if $|\lambda| < \frac{1}{M(b-a)}$

Note 2:

In equation (10), we rewrite the form as

$$k(x,\xi:\lambda) = k(x,\xi) + \lambda \sum_{n=0}^{\infty} \lambda^n k_{n+2}(x,\xi)$$

$$= k(x,\xi) + \lambda \sum_{n=0}^{\infty} \lambda^n \int_a^b k(x,\xi) k_{n+1}(\xi_1,\xi)\, d\xi_1$$

$$= k(x,\xi) + \lambda \sum_{n=0}^{\infty} \lambda^n \int_a^b k(x,\xi) \Gamma(\xi_1,\xi:\lambda)\, d\xi_1$$

Or

Changing the notation $\xi_1 \to \xi, \xi \to t$

$$\Gamma(x,t:\lambda) = k(x,t) + \lambda \int_a^b k(x,\xi)\Gamma(\xi,t:\lambda) d\xi$$

Thus it follows that the resolvent kernel Γ, consider as a function of the variable x and t and the parameter λ, is the solution of the equation (8)

$$y(x) = F(x) + \lambda \int_a^b k(x,\xi) y(\xi) d\xi$$

When the prescribed function F is replaced by the kernel k consider at function of x and t

Problem

Find the resolvent kernel of $y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi$

Soln:

Consider,

$$y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi$$

Here $k(x,\xi) = 1 - 3x\xi = k_1(x,\xi)$

$$k_2(x,\xi) = \int_a^b k(x,\xi_1)\, k(\xi_1,\xi) d\xi_1$$

$$= \int_0^1 (1 - 3x\xi)\,(1 - 3\xi_1\xi) d\xi_1$$

$$= \int_0^1 (1 - 3\xi_1\xi - 3x\xi_1 + 9\xi\xi_1^2 x) d\xi_1$$

$$= \left[\xi - \frac{3\xi_1^2\xi}{2} - \frac{3x\xi_1^2}{2} + \frac{9\xi\xi_1^3 x}{3}\right]_0^1$$

$$= 1 - \frac{3\xi}{2} - \frac{3x}{2} + \frac{9\xi x}{3}$$

$$k_2(x,\xi) = 1 - \frac{3\xi}{2} - \frac{3x}{2} + 3x\xi$$

$$k_3(x,\xi) = \int_a^b k(x,\xi_1)\, k_2(\xi_1,\xi) d\xi_1$$

$$= \int_0^1 (1 - 3x\xi) \left(1 - \frac{3}{2}(\xi_1 + \xi) + 3\xi\xi_1\right) d\xi_1$$

$$= \int_0^1 \left(1 - \frac{3}{2}(\xi_1 + \xi) + 3\xi\xi_1 - 3x\xi_1 + \frac{9}{2}x\xi(\xi_1 + \xi) + 9x\xi_1\xi_1^2\right) d\xi_1$$

$$= \left[\xi_1 - \frac{3}{2}\frac{\xi_1^2}{2} - \frac{3}{2}\xi\xi_1 + \frac{3}{2}\xi_1^2\xi - \frac{3}{2}x\xi_1^2 + \frac{9}{2*3}\xi_1^3 x + \frac{9}{2}\frac{x\xi\xi_1^2}{2} - \frac{9x\xi\xi_1^2}{3}\right]_0^1$$

$$= 1 - \frac{3}{4} - \frac{3}{2}\xi + \frac{3}{2}\xi - \frac{3x}{2} + \frac{9x}{6} + \frac{9}{2}\frac{x\xi}{2} - \frac{9x\xi}{3}$$

$$= \frac{1}{4} - \frac{3x\xi}{4}$$

$$= \frac{1}{4}(1 - 3x\xi)$$

$$k_3(x, \xi) = \frac{k_1(x, \xi)}{4}$$

$$k_4(x, \xi) = \frac{k_2(x, \xi)}{4}$$

$$k_5(x, \xi) = \frac{k_3(x, \xi)}{4}$$

$$= \frac{1}{16}k_1(x, \xi)$$

$$\vdots$$

$$k_n = \frac{k_{n-2}}{4} \; for \; n \geq 3$$

$$\Gamma(x, \xi : \lambda) = k(x, \xi) + \lambda \sum_{n=0}^{\infty} \lambda^n k_{n+2}(x, \xi)$$

$$= k_1(x, \xi) + \lambda[k_2(x, \xi) + \lambda k_3(x, \xi) + \lambda^2 k_4(x, \xi) + \lambda^3 k_5(x, \xi) + \lambda^4 k_6(x, \xi) \ldots]$$

$$= k_1 + \lambda k_2 + \lambda^2 k_3 + \lambda^3 k_4 + \lambda^4 k_5 + \lambda^5 k_6 \ldots$$

$$= k_1 + \lambda k_2 + \frac{\lambda^2 k_1}{4} + \frac{\lambda^3 k_2}{4} + \frac{\lambda^4 k_1}{16} + \frac{\lambda^5 k_2}{12} + \cdots$$

$$= k_1\left(1+\frac{\lambda^2}{4}+\frac{\lambda^4}{16}+\cdots\right)+k_2\lambda\left(1+\frac{\lambda^2}{4}+\frac{\lambda^4}{16}+\cdots\right)$$

$$= \left(1+\frac{\lambda^2}{4}+\frac{\lambda^4}{16}+\cdots\right)(k_1+k_2\lambda)$$

$$= \left(1+\frac{\lambda^2}{4}+\frac{\lambda^4}{16}+\cdots\right)\left(1-3x\xi+\lambda\left(1-\frac{3}{2}(x+\xi)+3x\xi\right)\right)$$

$$= \left(1-\frac{\lambda^2}{4}\right)^{-1}\left[1-3x\xi+\lambda-\frac{3}{2}\lambda(x+\xi)+3\lambda x\xi\right]$$

$$= \frac{1}{\left(1-\frac{\lambda^2}{4}\right)}\left(1+\lambda-\frac{3}{2}\lambda(x+\xi)-3x\xi(1-\lambda)\right)$$

$$= \Gamma(x,\xi:\lambda)$$

The result is correct for values λ except $\lambda = \pm 2$. Resolvent kernel is correctly given by (2) for all such values of λ. However the series (1) converges only when $|\ \lambda\ | < 2$. We are able to sum series (1) and that the resultant function correctly.